Assignment No. 8

[1] The inter-arrival time of vehicles on a highway section is expressed by an exponential distribution. If the probability of inter-arrival time of vehicles being less than 30 seconds is 20% then find the inter-arrival time (in seconds) having cumulative probability of 90%?

a. 230.50  
b. 309.56  
c. 134.44  
d. 358.75

Answer: b) 309.56

[2] The annual highest streamflow at a gaging station in river is expected to follow log-normal distribution. If the probability that annual highest streamflow is between 800cumecs and 2500cumecs is 85% and annual highest streamflow with 20 year return period is 2500cumecs, then find the mean annual highest streamflow at the gaging site in cumecs?

a. 1421.36  
b. 2201.29  
c. 1274.53  
d. 1652.76

Answer: a) 1421.36

[3] The compressive strength of concrete cubes (say $X$) prepared under some laboratory condition are found to follow pdf of following form:

$$f_X(x) = \frac{C}{\sqrt{2\pi}} e^{-(C(x-\mu))^2} \quad x > 0$$

If for seven cubes tested, the compressive strength is given as 23.6 MPa, 24.1 MPa, 25.7MPa, 23.5 MPa, 24.4MPa, 25.9MPa and 26.1 MPa. Determine the parameter $C$ by the maximum likelihood method?

a. 0.236  
b. 0.528  
c. 0.373  
d. 0.684

Answer: d) 0.684

[4-1] Thirty-two concrete cubes were prepared under a certain condition. The sample mean of compressive strength of these cubes is found to be 22 MPa. If the standard
deviation is known to be 4 MPa, determine the 99% confidence interval of the mean strength of the concrete cubes in MPa?

\[ \text{Answer: b) (20.18, 23.82)} \]

[4-2] In the previous question, determine the 95% confidence interval of the mean strength of the concrete cubes?

\[ \text{Answer: c) (20.61, 23.39)} \]

[4-3] In the previous question find the characteristic strength of concrete (95% dependable strength)?

\[ \text{Answer: c) 15.42} \]

[5-1] Twelve concrete cubes prepared under an experiment have mean compressive strength of 30 MPa and standard deviation of their compressive strength is 4 MPa, determine the 99% confidence interval of the mean strength of the concrete?

\[ \text{Answer: c) (26.414, 33.586)} \]

[5-2] In the previous question, determine the 90% confidence interval of the mean strength of the concrete cubes?

\[ \text{Answer: b) (27.926, 32.074)} \]
c. (26.414, 33.586)
d. (29.050, 30.950)
Answer: b) (27.926, 32.074)

[6] The daily dissolved oxygen (DO) concentration at a particular location on a stream has been recorded for 25 days. The sample variance is found to be \( s^2 = 3.5 \text{ (mg/L)}^2 \). What is the 95% upper confidence limit of the population variance \( \sigma^2 \)?

a. 8.850
b. 5.187
c. 13.850
d. 6.065

Answer: d) 6.065

[7] The streamflow at a gauge point for a river is recorded daily for the month of February, 2015. The standard deviation of streamflow is found to be 12 cumecs. Assuming that historically the streamflow of the river in month of February follows normal distribution, what is the 95% upper confidence limit of the long term streamflow variance \( \sigma^2 \) in the month?

a. 256.15
b. 240.74
c. 266.82
d. 223.56

Answer: b) 240.74

**Answers**

[1] Let \( T \) be the random variable for inter-arrival time and \( t_v \) be the average inter-arrival time between the vehicles, then CDF is given by

\[
F_T(t) = 1 - e^{-\frac{t}{t_v}}
\]

According to question,

\[
F_T(30) = 0.2
\]

\[
1 - e^{-\frac{30}{t_v}} = 0.2
\]

\[
\Rightarrow t_v = \frac{-30}{\ln(1 - 0.2)} = 134.44 \text{ seconds}
\]

Let \( t_{90} \) be the inter-arrival time having cumulative frequency of 90% then

\[
F_T(t_{90}) = 0.90
\]

\[
1 - e^{-\frac{t_{90}}{134.44}} = 0.9
\]

\[
\Rightarrow t_{90} = -(\ln(1 - 0.9))(134.44) = 309.56 \text{ seconds}
\]
Let $X$ be annual highest streamflow and $Y = \ln(X)$. So, $X$ follows log normal distribution but $Y$ follows normal distribution.

Let $\overline{X}$ and $C_v$ be mean and coefficient of variation of $X$ respectively.

Let $\overline{Y}$ and $S_Y$ be the mean and standard deviation of $Y$ respectively.

According to question the 20 year return period streamflow is 2500 cumecs.

$P(X > 2500) = \frac{1}{\sqrt{20}} = 0.05$

$\Rightarrow P(Y > \ln(2500)) = 0.05$

$\Rightarrow P(Y \leq 7.824) = 1 - 0.05$

$\Rightarrow P\left( Z \leq \frac{7.824 - \overline{Y}}{S_Y} \right) = 0.95$

$\Rightarrow \frac{7.824 - \overline{Y}}{S_Y} = 1.645 \quad \text{eq1}$

and, $P(800 \leq X \leq 2500) = 0.85$

Hence,

$P(X < 800) = 1 - P(X \geq 800) = 1 - P(800 \leq X \leq 2500) - P(X > 2500) = 0.1$

$\Rightarrow P(Y < \ln(800)) = 0.1$

$\Rightarrow P(Y < 6.6846) = 0.1$

$\Rightarrow P\left( Z < \frac{6.6846 - \overline{Y}}{S_Y} \right) = 0.1$

$\Rightarrow \frac{6.6846 - \overline{Y}}{S_Y} = -1.282 \quad \text{eq2}$

By solving eq2 and eq1 simultaneously,

$\overline{Y} = 7.1836 \quad \text{and} \quad S_Y = 0.3893$

$\Rightarrow \frac{1}{2} \ln\left( \frac{\overline{X}^2}{1 + C_v^2} \right) = 7.1836 \quad \text{and} \quad \sqrt{\ln(1 + C_v^2)} = 0.3893$

$\Rightarrow \overline{X} = 1421.358 \quad \text{and} \quad C_v = 0.4045$

So, Mean annual highest streamflow = 1421.36 cumecs

[3] Here, $\mu = \overline{X} = \frac{1}{7} \sum_{i=1}^{7} x_i = 24.7571$ MPa

$f_x(x) = \frac{C}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} = \frac{C}{\sqrt{2\pi}} e^{-\frac{(x-24.7571)^2}{2}}$

$L(x_1, x_2, \ldots, x_7; \lambda) = \prod_{i=1}^{7} \frac{C}{\sqrt{2\pi}} e^{-\frac{(x_i-24.7571)^2}{2}}$

$= \left( \frac{C}{\sqrt{2\pi}} \right)^7 \exp\left( -C^2 \sum_{i=1}^{7} (x_i - 24.7571)^2 \right)$
\[
\frac{\partial L}{\partial C} = 7 \frac{C^6}{(2\pi)^{3.5}} \exp \left( -C^2 \sum_{i=1}^{7} (x_i - 24.7571)^2 \right) - \frac{2C^8 \sum_{i=1}^{7} (x_i - 24.7571)^2}{(2\pi)^{3.5}} \exp \left( -C^2 \sum_{i=1}^{7} (x_i - 24.7571)^2 \right)
\]

\[
\frac{\partial L}{\partial C} = 0
\]

or, \[7C^6 - 2C^8 \sum_{i=1}^{7} (x_i - 24.7571)^2 = 0\]

or, \[2C^2 \sum_{i=1}^{7} (x_i - 24.7571)^2 = 7\]

or, \[C^2 = \frac{3.5}{\sum_{i=1}^{7} (x_i - 24.7571)^2}\]

or, \[C = \sqrt{0.4681} = 0.684\]

[4-1] For the 99% confidence interval,
\[\alpha = 1 - 0.99 = 0.01\]

From the standard normal table,
\[P(Z \leq z_{\alpha/2}) = 1 - \frac{\alpha}{2}\]

or, \[P(Z \leq z_{0.005}) = 1 - 0.005\]

or, \[P(Z \leq z_{0.005}) = 0.995\]

or, \[z_{0.005} = 2.575\]

Now, \[\frac{\sigma}{\sqrt{n}} z_{\alpha/2} = \frac{4}{\sqrt{32}} (2.575) = 1.8208\]

The 99% confidence interval of the mean strength of the concrete cubes is
\[(22 - 1.8208; 22 + 1.8208) \text{ MPa}\]

i.e, \((20.18; 23.82) \text{ MPa}\)

[4-2] Similarly, for the 95% confidence interval
\[\alpha = 1 - 0.95 = 0.05\]

From the standard normal table,
\[ P(Z \leq z_{\alpha/2}) = 1 - \frac{\alpha}{2} \]

or, \[ P(Z \leq z_{0.025}) = 1 - 0.025 \]

or, \[ P(Z \leq z_{0.025}) = 0.975 \]

or, \[ z_{0.025} = 1.96 \]

Now, \[ \frac{\sigma}{\sqrt{n}} z_{\alpha/2} = \frac{4}{\sqrt{32}} (1.96) = 1.3859 \]

The 95% confidence interval of the mean strength of the concrete cubes is

\[ (22 - 1.3859; 22 + 1.3859) \text{ MPa} \]

\[ \text{i.e., } (20.61; 23.39) \text{ MPa} \]

[4-3] For the 95% dependable strength, the cumulative probability is 0.05, corresponding to which standard normal variate is -1.645.

So, characteristic strength of concrete = 22 - 4*1.645 = 15.420 MPa

[5-1] For the 99% confidence interval,

\[ \alpha = 1 - 0.99 = 0.01 \]

As sample size is small, so t-distribution is used. Degree of freedom for t-distribution is n-1 i.e. 11 in this case. From the t-distribution table,

\[ P(T \leq t_{\alpha/2,11}) = 1 - \frac{\alpha}{2} \]

or, \[ P(T \leq t_{0.005,11}) = 1 - 0.005 \]

or, \[ P(T \leq t_{0.005,11}) = 0.995 \]

or, \[ t_{0.005,11} = 3.1058 \]

Now, \[ \frac{\sigma}{\sqrt{n}} t_{\alpha/2,11} = \frac{4}{\sqrt{12}} (3.1058) = 3.5863 \]

The 99% confidence interval of the mean strength of the concrete cubes is

\[ (30 - 3.5863; 30 + 3.5863) \text{ MPa} \]

\[ \text{i.e., } (26.414; 33.586) \text{ MPa} \]

[5-2] Similarly, for the 90% confidence interval, from t-distribution table,
\[ P(T \leq t_{\alpha/2,11}) = 1 - \frac{\alpha}{2} \]

or, \[ P(T \leq t_{0.05,11}) = 1 - 0.05 \]

or, \[ P(T \leq t_{0.05,11}) = 0.95 \]

or, \[ t_{0.05,11} = 1.7959 \]

Now, \[ \frac{\sigma}{\sqrt{n}} t_{\alpha/2,19} = \frac{4}{\sqrt{12}} (1.7959) = 2.0737 \]

The 90% confidence interval of the mean strength of the concrete cubes is

\[ (30 - 2.0737; \quad 30 + 2.0737) \text{ MPa} \]

i.e, \( (27.926; \quad 32.074) \text{ MPa} \)

[6] Sample size (n) = 25, Level of significance (\(\alpha\)) = 1 - 0.95 = 0.05

\( (n-1)= 24 \), from chi square tale

\[ \chi^2_{0.05,24} = 13.85 \]

Hence, the 95% upper confidence limit of the population variance

\[ \frac{(n - 1)s^2}{\chi^2_{0.05,24}} = \frac{24 \times 3.5}{13.85} = 6.065 \text{ (mg / L)}^2 \]

[7] Sample size (n) = Number of days in February, 2015 = 28

Level of significance (\(\alpha\)) = 1 - 0.95 = 0.05

\( (n-1)= 27 \), from chi square tale

\[ \chi^2_{0.05,27} = 16.15 \]

Hence, the 95% upper confidence limit of the long term streamflow variance

\[ \frac{(n - 1)s^2}{\chi^2_{0.05,27}} = \frac{27 \times (12)^2}{16.15} = 240.74 \text{ (m}^3 / \text{s)}^2 \]