Assignment 8

Due on 2023-03-10, 23:59 EST

Problem 1: Transient Heat Conduction

\[
\frac{\partial u}{\partial t} = \alpha \nabla^2 u + g(x,t)
\]

where \(u\) is the temperature, \(\alpha\) is the thermal diffusivity, \(g(x,t)\) is the heat source, and \(\nabla^2\) is the Laplacian operator. We will solve the following transient heat conduction problem:

**Problem:**

A metal rod of length 1 m is initially at a uniform temperature of 20°C. At time \(t = 0\), a heat source is applied to the right end of the rod, such that \(g(x,t) = 100\text{ W/m}^2\) for \(x = 1\text{ m}\) and \(g(x,t) = 0\) for \(x < 1\text{ m}\). The initial temperature distribution is \(u(x,0) = 20\text{°C}\). The thermal diffusivity of the metal is \(\alpha = 0.1\text{ m}^2/\text{s}\).

**Solution:**

1. Derive the governing equation for this problem.
2. Apply the appropriate boundary and initial conditions.
3. Set up the finite difference equation and solve for the temperature distribution at each time step.
4. Plot the temperature profile at a selected time step.

Problem 2: Mass-Spring-Damper System

Consider a mass-spring-damper system described by the following equations:

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -c v - k x + f(t)
\end{align*}
\]

where \(x\) is the displacement, \(v\) is the velocity, \(c\) is the damping coefficient, \(k\) is the spring constant, and \(f(t)\) is the external force. Given \(c = 2\), \(k = 4\), and \(f(t) = 10\cos(2t)\), find the steady-state response of the system.

**Solution:**

1. Derive the differential equation for the system.
2. Solve for the steady-state response of the system.
3. Plot the steady-state response over a selected time interval.

Problem 3: Reactor In Three Variables

A chemical reactor is described by the following set of coupled ODEs:

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(x_1, x_2, x_3) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, x_3) \\
\frac{dx_3}{dt} &= f_3(x_1, x_2, x_3)
\end{align*}
\]

where \(x_1, x_2, x_3\) are the state variables. The functions \(f_1, f_2, f_3\) are given by:

\[
\begin{align*}
f_1(x_1, x_2, x_3) &= 2x_1 - x_2 + x_3 \\
f_2(x_1, x_2, x_3) &= x_1^2 - x_2^2 + x_3 \\
f_3(x_1, x_2, x_3) &= x_1 + x_2 - 2x_3
\end{align*}
\]

**Solution:**

1. Derive the system of ODEs for the reactor.
2. Solve the system numerically for a given initial condition.
3. Plot the state variables over a selected time interval.

Problem 4: Predator-Prey Model

This model describes the interaction between two species, one of which preys on the other. The model is given by:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1(1 - x_1) - x_1 x_2 \\
\frac{dx_2}{dt} &= x_2(1 - x_2) + \frac{r x_1 x_2}{x_1 + \alpha}
\end{align*}
\]

where \(x_1\) and \(x_2\) are the population sizes of the prey and predator, respectively, and \(r, \alpha\) are positive constants. Given \(r = 2, \alpha = 1\), find the equilibrium points and their stability.

**Solution:**

1. Derive the system of ODEs for the predator-prey model.
2. Find the equilibrium points for the system.
3. Analyze the stability of the equilibrium points.
4. Plot the phase portrait of the system over a selected time interval.