46.1 Line integrals

In this section we describe a natural generalization of the notion of definite integral, called the line integral. This notion finds many applications.

46.1.1 Definition:

Let

\[ f : D \subset \mathbb{R}^3 \to \mathbb{R} \]

and \( C \) be a curve in \( \mathbb{R}^3 \) with parameterization

\[ r : [a,b] \to \mathbb{R}^3, \text{ where } r(t) \in D \text{ for } t \in [a,b] \]

Let \( r \) have the arc length parametrization \( r(s), a \leq s \leq b \). Then the function
is a scalar-valued continuous function on the interval \([a,b]\) for both \(f\) and \(r\) are continuous. Thus, the integral
\[
\int_c f ds := \int_{s=a}^{s=b} (r \cdot f)(s) \, ds
\]
is well-defined. It and is called the **line integral** of \(f\) over \(C\).

The line integral being a definite integral, has the following properties.

### 46.1.2 Theorem

If
\[
f : D \subset \mathbb{R}^3 \rightarrow \mathbb{R}
\]
is continuous and \(C\) is a simple, regular curve in \(D\) with a parameterization \(r(t), t \in [c, d]\), then
\[
\int_C f \, ds = \int_c^d f(x(t), y(t), z(t)) \| r'(t) \| \, dt.
\]

**Proof**

Since
\[
\int_C f \, ds = \int_a^b f(x(s), y(s), z(s)) \, ds,
\]
and
\[
\frac{ds}{dt} = \| r'(t) \|,
\]
from (1) and (2) we have
\[
\int_C f \, ds = \int_a^b f(x(t), y(t), z(t)) \| r'(t) \| \, dt.
\]

### 46.1.3 Note:

In defining \(\int_C f \, ds\), implicitly we have assumed that the arc length increases as the variable increases. This is normally, called the **positive orientation** on \(C\). The opposite orientation will give a change of sign for \(\int_C f \, ds\).

### 46.1.4 Examples:

1. Let us evaluate
\[
\int_C (1 + xy^2) \, ds,
\]
where \(C\) is the line segment from \((0,0)\) to \((1,2)\) in \(\mathbb{R}^2\).
To move from \((0,0)\) to \((1,2)\), let us choose the parameterization
\[
\mathbf{r}(t) = t \mathbf{i} + 2t \mathbf{j}, \quad 0 \leq t \leq 1.
\]
Then,
\[
||\mathbf{r}'(t)|| = \sqrt{5}.
\]
Hence,
\[
\int_{C}(1+xy^2)\,ds = \int_{0}^{1} (1+4t^2)\sqrt{5}\,dt = 2\sqrt{5}.
\]

2. Let us calculate
\[
\int_{C} f\,ds \quad \text{for} \quad f(x,y,z) = xy + z^3
\]
where \(C\) is the circular helix
\[
\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}, \quad \text{from} \ (1,0,0) \ \text{to} \ (-1,0,\pi).
\]
Note that, to move from \((1,0,0)\) to \((-1,0,\pi)\) along \(\mathbf{r}(t), t\) varies our \([0,\pi]\).
Since
\[
||\mathbf{r}'(t)|| = \sqrt{1+1} = \sqrt{2}, \quad 0 \leq t \leq \pi,
\]
we have
\[ \int_C f \, ds = \sqrt{2} \int_0^\pi (\cos t \sin t + t^3) \, dt \]
\[ = \sqrt{2} \left[ \frac{\sin^2 t}{2} + \frac{t^4}{4} \right]_0^\pi \]
\[ = \frac{\sqrt{2} \pi^4}{4}. \]

46.1.5 **Theorem (Properties of the line integral):**

1. \[ \int_C (f + g) \, ds = \int_C f \, ds + \int_C g \, ds \]
2. \[ \int_C (\alpha f) \, ds = \alpha \left( \int_C f \, ds \right) \]
3. If \( C \) consists of finite number of pieces \( C_1, C_2, \ldots, C_n \), where each \( C_i \) is regular, then

\[ \int_C f \, ds = \sum_{i=1}^n \left( \int_{C_i} f \, ds \right). \]

**Proof**

We assume these properties.

46.1.6 **Example:**

Let us compute

\[ \int_C f \, ds \]

where \( f(x, y, z) = x + \sqrt{y - z^2} \),

and the path \( C \) given by

\( y = x^2 \) from \( O(0, 0, 0) \) to \( A(1, 1, 0) \) and the line segment from \( A(1, 1, 0) \) to \( B(1, 1, 1) \).

**Figure:** The curve \( C = C_1 \cup C_2 \)

We can think of \( C \) as two pieces, \( C_1 \) from \( O \) to \( A \) along \( y = x^2 \) with parameterization given by

\[ r(t) = ti + t^2j, \quad 0 \leq t \leq 1, \]

and the piece \( C_2 \) the path from \( A \) to \( B \) along the line segment joining them, with parameterization given by

\[ r(t) = i + j + tk, \quad 0 \leq t \leq 1. \]

Thus, the curve \( C \) consists of two pieces \( C_1 \) and \( C_2 \), both of which are regular, and hence
46.1.7 Definition:

Let $C$ be a smooth parametric curve with a parameterization $\mathbf{r}(t), t \in [a, b]$. Consider the curve

$$\tilde{\mathbf{r}}(t) := \mathbf{r}(b - (t - a)), t \in [a, b]$$

Then, $\tilde{\mathbf{r}}$ is also a smooth curve. Geometrically,

$$([\tilde{\mathbf{r}}(t)]t \in [a, b]) = ([\mathbf{r}(t)]t \in [a, b]).$$

However, $\tilde{\mathbf{r}}$ traverses the path $C$ backwards, i.e., the initial-point of $\mathbf{r}$ is the final-point of $\tilde{\mathbf{r}}$ and vice-verse. The curve $\tilde{\mathbf{r}}$ is called the reverse of $C$, and is denoted by $-C$.

46.1.8 Theorem:

Let $f : D \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ be a scalar field. If $\int_C f \, ds$ exists, then $\int_{-C} f \, ds$ also exists and

$$\int_{-C} f \, ds = -\left\{ \int_C f \, ds \right\}.$$

**Proof**

Follows from the fact that

$$\frac{d\tilde{\mathbf{r}}(t)}{dt} = -\frac{d\mathbf{r}(t)}{dt}, \text{ for every } t \in [a, b].$$

46.1.9 Definition:

Let $C$ be a smooth curve in $D \subseteq \mathbb{R}^3$ with parameterization

$$\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}, t \in [a, b]$$

1. For a continuous scalar field $f : D \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$, define

$$\int_C f \, dx := \int_{t=a}^{t=b} f(x(t), y(t), z(t)) \left( \frac{dx}{dt} \right) \, dt,$$

$$\int_C f \, dy := \int_{t=a}^{t=b} f(x(t), y(t), z(t)) \left( \frac{dy}{dt} \right) \, dt,$$
2. For a continuous vector field $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}$, with

$$\mathbf{F} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k},$$

define

$$\int_C \mathbf{F} \cdot d\mathbf{r} := \int_C f_1 \, dx + \int_C f_2 \, dy + \int_C f_3 \, dz,$$

Called the **line integral** of $\mathbf{F}$ over $C$.

46.1.10 **Note:**

The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends not both upon the orientation (positive or reverse) of $C$, also upon the initial and the final points of $C$.

46.1.11 **Example:**

Let $C_1$ and $C_2$ be smooth curves given by

$$\mathbf{r}_1(t) := t \mathbf{i} + t \mathbf{j} + t \mathbf{k}, 0 \leq t \leq 1$$

and

$$\mathbf{r}_2(t) := t \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}, 0 \leq t \leq 1.$$ 

Then, $C_1$ and $C_2$ both have initial point $(0,0,0)$ and final point $(1,1,1)$. Further, for the vector field $\mathbf{F} := yz \mathbf{i} + xz \mathbf{j} + x^2 y \mathbf{k}$,

we have

$$\int_{C_1} yz \, dx + xz \, dy + yx^2 \, dz = \int_0^1 \left( t^2 \, dt + t^2 \, dt + t^3 \, dt \right)$$

$$= \left[ \frac{2}{3} t^3 \right]_0^1 + \left[ \frac{t^4}{4} \right]_0^1 = \frac{11}{12}.$$ 

And

$$\int_{C_2} yz \, dx + xz \, dy + yx^2 \, dz = \int_0^1 \left( t^4 \, dt + t^4 \left( 2t \, dt \right) + t^4 \left( 3t^2 \, dt \right) \right)$$

$$= \left[ \frac{t^5}{5} \right]_0^1 + \left[ \frac{2t^5}{5} \right]_0^1 + \left[ \frac{3t^6}{6} \right]_0^1 = \frac{11}{10}.$$ 

**For Quiz refer the WebSite**

**Practice Exercises**

Evaluate the following line integrals:

1. $\int_C \left( x^2 - y + 3z \right) \, ds$, where $C$ is the line segment going $(0,0,0)$ with $(1,2,1)$. 

2. \[ \int_C \frac{1}{1+x^2} \, ds \], where \( C \) is the curve \( r(t) = t \, \textbf{i} + \frac{3}{2} t^3 \, \textbf{j}, \, 0 \leq t \leq 3 \):

**Answer:** 2

3. For the given vector field \( \textbf{F} \) and the curve \( C \), compute \( \int_C \textbf{F} \cdot d\textbf{r} \):

\[ \textbf{F}(x,y) = x^2 \, \textbf{i} + xy \, \textbf{j}, \]
\( C \) is the circle \( x^2 + y^2 = 4 \).

\[ \textbf{F}(x,y,z) = x^2 y \, \textbf{i} + (x-z) \, \textbf{j} + xyz \, \textbf{k}, \]
\( C \) is the curve \( r(t) = t \, \textbf{i} + t^2 \, \textbf{j} + 2 \, \textbf{k}, \, 0 \leq t \leq 1 \)

**Answer:** 0

- \(-\frac{17}{13}\)

4. Let \( f(x,y) = x - y \) and \( C \) be the curve \( r(t) = 2 \, t \, \textbf{i} + 3 t^2 \, \textbf{j} \). Compute the following: \( \int_C f \, dx, \int_C f \, dy \)

**Answer:** 0, \(-\frac{1}{2}\)

5. Compute the following:

1. \[ \int_C y \, dx + x^2 \, dy \],
   where \( C \) is the arc of the parabola \( y = 4x - x^2 \) from \((4,0)\) to \((1,3)\).

2. \[ \int_C zy \, dx + x^2 \, dy + xy \, dz \]
   where \( C \) is the curve \( r(t) = e^t \, \textbf{i} + e^t \, \textbf{j} + e^t \, \textbf{k} \) from \( 0 \leq t \leq 1 \)

**Answer:** \( \frac{9}{2} \)

- \( 1 - e^3 \)

**Recap**

In this section you have learnt the following:

- How to define the integrals of a scalar field over a curve.

[**Section 46.2**]

**Objectives**

In this section you will learn the following:

- How to use line integral to compute areas of some surfaces.
- Physical applications of line integrals.
46.2 Applications of line integral

46.2.1 Surface area of a thin sheet:

Suppose we have a surfaces $S$ whose base is a curve $C$ in the $xy$-plane and its height at any point $(x,y) \in C$ in the value $z = f(x,y)$, where $f$ is some function which domain includes $C$.

![Surface with base C and height z = f(x,y)](image)

Figure 181. Surface with base $C$ and height $z = f(x,y)$

We can think of this surface as made up of small vertical strips with base $\Delta s$ and height $f(x,y)$. The area of this strip is approximately given by $f(x,y)\Delta s$. Thus, the total area of this surface can be defined to be

$$\lim_{\Delta s \to 0} \left( \sum_{\Delta s} f(x,y)\Delta s \right),$$

whenever it exists. This limit is nothing but $\int_C f ds$. Thus, the area of the surface $S$ can be defined to be
Area of $S := \int_{C} f \, ds$.

**46.2.2 Example:**

Let us compute the area of the surface 
$S$ with base the circle $x^2 + y^2 = 1$ in the $xy$-plane 
extending upward to 
the parabolic cylinder $z = 1 - x^2$ at the top. 
The required area is given by 
$$A = \int_{C} (1 - x^2) \, ds,$$
where $C$ is the circle with arc-length parameterization :
$$r(s) = \cos s \mathbf{i} + \sin s \mathbf{j}, \quad 0 \leq s \leq 2\pi.$$

Figure: Surface with base the circle and height $z = 1 - x^2$

Thus, 
$$A = \int_{0}^{2\pi} \pi (1 - \cos^2 s) \, ds = \int_{0}^{2\pi} \pi \sin s \, ds = \pi$$

**46.2.3 Mass and Center of gravity of a thin wire :**

Consider a thin wire in the shape of a curve $C$ in space.

Figure: A piece of wire $C$

If $f(x, y, z)$ represents the mass per unit length of the wire, then the mass of a small portion $\Delta s$ of the wire, is given by
Thus, we can define the total mass of the wire to be

$$\Delta M := f(x, y, z)\Delta s.$$  

Similarly, we can define the moments of the wire $C'$ about the coordinate planes as follows

$$M_{xy} := \int_C zf(x,y,z)\,ds, \quad M_{xz} := \int_C xf(x,y,z)\,ds, \quad M_{yz} := \int_C yf(x,y,z)\,ds.$$  

Finally, the point $\left(\bar{x}, \bar{y}, \bar{z}\right)$, called the center of mass of the wire, is defined by

$$\bar{x} := \frac{M_{yz}}{M}, \quad \bar{y} := \frac{M_{zx}}{M}, \quad \bar{z} := \frac{M_{xy}}{M}.$$  

### 46.2.4 Work done along a curve:

Consider a force $\mathbf{F}$ being applied to a body to move it along a curve $C$ from a point $A$ to a point $B$.

![Figure: 177](image)

If $\mathbf{r}(t), t_0 \leq t \leq t_1$, is a parameterization of $C$, then the amount of work done to move the body by a small distance $\Delta s$ along the curve is given by

$$\left(\mathbf{F} \cdot \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}\right)\Delta s,$$

since $\mathbf{F} \cdot \mathbf{r}'(t)$ is the tangential component of force. Thus $W$, the total work done in moving the body along $C$, is given by

$$W = \int_C \left(\mathbf{F} \cdot \frac{\mathbf{r}'}{||\mathbf{r}'||}\right)\,ds = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right)\,ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=t_0}^{t_1} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right)\,dt.$$  

---------- (37)

If $\mathbf{F}$ has components $F_1, F_2, \text{and} F_3$, i.e.,

$$\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}, \quad \text{and} \quad \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k},$$

then equation (37) can also be written as
46.2.5 Circulation of a fluid along a curve:

Let \( \mathbf{v} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \) be the velocity field of a fluid flowing through a region \( D \) in space. Let \( C \) be a curve inside the region \( D \).

Then the tangential component of \( \mathbf{v} \) at a point on the curve is given by

\[
W = \int_{t=t_0}^{t_1} \left( F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt
\]

Then, for a small portion \( \Delta s \) of the curve, the quantity \( (\mathbf{v} \cdot \mathbf{T}) \Delta s \) represents the flow of the fluid flow across the small portion \( \Delta s \). Thus, the total flow of the fluid along the curve \( C \) is given by

\[
\text{Total flow along } C = \int_C (\mathbf{v} \cdot \mathbf{T}) \, ds.
\]

If the curve \( C \) is a closed curve, then the above integral is called the circulation of the flow along the curve.

46.2.6 Flux across a plane curve:

Consider a fluid flowing in a region \( D \) in the plane. Let \( \mathbf{v} \) be the velocity vector of the fluid and \( \rho(x,y) \) be its density at a point \( (x,y) \in D \).

Then, the vector field

\[
\mathbf{F}(x,y) = \rho(x,y) \mathbf{v}(x,y), \quad (x,y) \in D
\]

represents the rate of change of mass, per unit time across a unit length. Let \( \Gamma \) be a curve in the domain \( \Omega \). Then the rate of
change of mass of the fluid across a small portion $\Delta s$ of the curve is given by 

$$(\mathbf{F} \cdot \mathbf{n}) \Delta s,$$

where $\mathbf{n}$ is the unit normal vector to the curve. Thus, the total mass flow across whole of $C$ is given by 

$$\text{Total flow across } C := \int_C (\mathbf{F} \cdot \mathbf{n}) \, ds,$$

called the flux of the fluid flow across $C$.

**Practice Exercises**

1. Compute the area of the surface with base on the curve $C$ in the $xy$-plane and at the point $(x,y)$ in $C$, the height being $z = f(x,y)$ for the following:
   1. $f(x,y) = xy$, $C$ is the part of the unit circle $x^2 + y^2 = 1$ from $(1,0)$ to $(0,1)$.
   2. $f(x,y) = 3x$, $C$ is the parabola $y = x^3$, $0 \leq x \leq 2$.
   3. $f(x,y) = 2 + \frac{1}{2}(3y - 4y^2)$, $C$ is the unit circle.

   **Answer:**
   (i) $\frac{1}{2}$
   (ii) $\frac{17\sqrt{17} - 1}{4}$
   (iii) $4\pi$

2. Find the work done by a force field $\mathbf{F}(x,y,z)$, moving along a curve $C$ given below:
   1. $F(x,y,z) = \frac{x}{2} \mathbf{i} - \frac{y}{2} \mathbf{j} + \frac{z}{4} \mathbf{k}$, $C$ is $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$, $0 \leq t \leq 3\pi$.
   2. $F(x,y,z) = x \mathbf{i} + y \mathbf{j}$, $C$ is $\mathbf{r}(t) = 3t^2 \mathbf{i} + t \mathbf{j} + \mathbf{k}$, $0 \leq t \leq 1$.

   **Answer:**
   (i) $3\pi / 4$
   (ii) $5$

3. Find the circulation of $\mathbf{v}(x^2 + y^2) \,(\mathbf{i} + \mathbf{j})$ along the curve $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$, $0 \leq t \leq 2\pi$

   **Answer:** $0$

4. Find the flux of the vector field $\mathbf{F}(x,y) = y^3 \mathbf{i} + x^2 \mathbf{j}$ across the boundary of the unit square $[0,1] \times [0,1]$

   **Answer:** $0$

**Recap**

In this section you have learnt the following:

- How to use line integral to compute areas of some surfaces.
- Physical applications of line integrals.