39.1 Absolute maxima/minima

39.1.1 Definition:
Let \( f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \). If there exists a point \( a \in D \) such that
\[
f(x,y) \leq f(a), \text{ for all } (x,y) \in D,
\]
then the number \( f(a) \) is called the absolute maxima of \( f \) in \( D \). Similarly, if there exists a point \( b \in D \) such that
\[
f(x,y) \geq f(b), \text{ for all } (x,y) \in D,
\]
then the number \( f(b) \) is called the absolute minima of \( f \) in \( D \).

39.1.2 Note:
Recall that if \( D \) is closed and bounded, and \( f \) is continuous, then by theorem 30.2.4, both absolute maxima and absolute minimum exist.

39.1.3 Theorem:
Let \( f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \).
Let $f$ assume its absolute maximum at a point $a \in D$. Then, either at $a$ is a boundary point of $D$, or is a critical point of $f$ in $D$.

(i)

Let $f$ its absolute minimum at a point $b \in D$. Then, either $b$ is a boundary point of $D$, or is a critical point of $f$ in $D$.

(ii)

39.1.3 Theorem:

Let $f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$.

(i) Let $f$ assume its absolute maximum at a point $a \in D$. Then, either $a$ is a boundary point of $D$, or is a critical point of $f$ in $D$.

(ii) Let $f$ its absolute minimum at a point $b \in D$. Then, either $b$ is a boundary point of $D$, or is a critical point of $f$ in $D$.

Proof

Suppose $a$ is not a boundary point of $D$. Then, $f$ must assume its maximum at some interior point of $D$. Thus, $a \in D$ is an interior point of $D$. If both $f_x(a)$ and $f_y(a)$ exist, then $\nabla f(a) = 0$ by theorem 37.1.3. In any case, $a$ is a critical point of $f$. Similar arguments hold for $b$.

39.1.4 Note:

To find the absolute maximum $M$ and the absolute minimum $m$ of a function $f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ on $D$, we compare the values of $f$ at the critical points of $f$ in $D$ and the absolute maximum and the absolute minimum of the restriction of $f$ to the boundary of $D$. The latter can often be found by reducing it to a one variable problem.

39.1.5 Examples:

(i) Suppose

$$D = \{(x,y) \in \mathbb{R}^2 \mid |x| \leq 2, |y| \leq 2\}$$

and $f : D \rightarrow \mathbb{R}$ is given by

$$f(x,y) = 4xy - 2x^2 - y^4.$$ 

Since $D$ is a closed bounded set and $f$ is a continuous function, it has both, absolute maximum and absolute minimum in $D$. For $f$ both the partial derivatives exist everywhere and

$$f_x(x_0,y_0) = 4y_0 - 4x_0, f_y(x_0,y_0) = 4x_0 - 4y_0.$$ 

Further,
\[ \nabla f(x_0, y_0) = 0 \text{ for } (x_0, y_0) = (0, 0), (1, 1) \text{ and } (-1, -1). \]

Also, \((x_0, y_0) \in D\) is a boundary point if
\[ x_0 = 2 \text{ or } x_0 = -2 \text{ or } y_0 = 2 \text{ or } y_0 = -2. \]

Due to symmetry of the domain, \(f(-x, -y) = f(x, y)\). Thus, we need only determine the absolute maximum and minimum of the functions
\[ f(2, y) = 8y - 8y^4, \text{ for } -2 \leq y \leq 2 \]
and
\[ f(x, 2) = 8x - 2x^2 - 16, \text{ for } -2 \leq x \leq 2. \]

It is easy to check that the function
\[ f(2, y) \text{ has absolute maximum at } y = \frac{\sqrt{2}}{2} \]
and
\[ \text{absolute minimum at } y = -2. \]

Also,
\[ f(x, 2), \text{ absolute maximum at } x = 2 \text{ and absolute minimum at } x = -2. \]

Finally, we compare these values of \(f\)
\[ f(0, 0) = 0, f(1, 1) = 1, f(2, \sqrt{2}) = 6\sqrt{2} - 8, f(2, -2) = -40, f(2, 2) = -8, \]
here we have ignored the points \((-1, -1)\) and \((-2, 2)\) due to symmetry. Thus,
the absolute maximum of \(f\) is 1,
which is attained at the points \((1, 1)\) as well as at \((-1, -1)\), and
the absolute minimum of \(f\) is \(-40, \)
which is attained at \((2, -2)\) as well as at \((-2, 2)\).

(ii) Let us find the triangle for which the product of the sines of the three angles is the largest. If we denote two
angles by \(x\) and \(y\), then the required function to be maximized is
\[ f(x, y) = \sin x \sin y \sin(x + y), \text{ where } 0 \leq x, y, x + y \leq \pi. \]

It is obvious from the nature of the problem that the function will have absolute maximum. Note that
\[ f(x, y) = 0 \text{ if } x = y \text{ or } x + y = 0, \text{ or } \pi. \]

Thus, \(f\) vanishes at each boundary point. At other points, i.e., for \(0 < x, y, x + y < \pi\), the equations
\[ f_x(x, y) = 0 = f_y(x, y) \text{ are given by} \]

\[ \frac{\sin x \sin (x + y) - \sin x \sin x \sin y}{\sin(x + y) - \sin x} = \frac{\sin y \sin (x + y) - \sin x \sin y \sin x}{\sin(x + y) - \sin y} \]
Since 
\[ \sin x \neq 0 \quad \text{and} \quad \sin y \neq 0 \quad \text{for} \quad 0 < x, y, < \pi, \]
above equations give us 
\[ \sin(2x + y) = 0 = \sin(2y + x). \]

As \( 0 < x + y < \pi, \) we have \( 0 < 2x + y, 2y + x < 2\pi, \) and hence, the critical points of \( f \) are given by
\[ 2x + y = 0 = 2y + x, \quad \text{i.e.,} \quad x = y = \frac{\pi}{3}. \]

Since \( f \left( \sqrt{3}, \sqrt{3} \right) > 0, \) follows that
\( f \) has an absolute maximum at \( \left( \sqrt{3}, \sqrt{3} \right). \)
Thus, the desired triangle is equilateral.

**Practice Exercises**

(1) Find the absolute minimum and the absolute maximum of the function

\[ f(x, y) = 2x^2 - 4x + y^2 - 4y + 1 \]

on the closed triangular plate bounded by the lines

\( x = 0, y = 2 \) and \( y = 2x. \)

**Answer**

(2) Find the absolute maximum and the absolute minimum of

\[ f(x, y) = (x^2 - 4x) \cos y \]

over the region

\( R = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 3, -\pi/4 \leq y \leq \pi/4 \}. \)

**Answer**

**Recap**

In this section you have learnt the following

- The notion of absolute maxima/minima for functions of several variables.
- Method of finding absolute maxima/minima.

**Objectives**
In this section you will learn the following:

- The notion of constrained maxima/minima for functions of several variables.
- Lagrange's Method of finding constrained maxima/minima.

39.2 Constrained maxima/minima

In many practical problems, one has to find extreme values of a function whose domain is constrained to lie on a particular region in space. Let us look at some examples.

39.2.1 Examples:

(i) Find the point \( P(x, y, z) \) closest to the origin on the plane \( 2x + y - 3z = 5 \).

Here we want to find \((x, y, z)\) in the plane \( 2x + y - 3z = 5 \) such that the value

\[
f(x, y, z) = \sqrt{x^2 + y^2 + z^2}
\]

is the smallest.

(ii) A space satellite in the shape of an ellipsoid \( 4x^2 + y^2 + 4z^2 = 16 \) enters earth's atmosphere and its
surface begins to heat. After an hour, the temperature at the point \((x, y, z)\) on the surface of the probe is
\[
T(x, y, z) = 8x^2 + 3yz - 16z + 600.
\]

One would like to find the hottest point on the surface of the probe. That leads to the problem of finding absolute maximum \(T\) for \((x, y, z)\) on the ellipsoid.

Mathematically, the problem is to find the absolute maximum/minimum of a function
\[
f : D \to \mathbb{R}, \text{ where } (x, y) \text{ is constrained to satisfy } g(x, y) = 0.
\]

In case we can solve \(g(x, y) = 0\) for one of the variables in terms of the other, the problem can be reduced to a problem of one variable. But, often this is difficult. A method to handle such problems, without having to solve the constraint equation and giving preference to one of the variables. This method is based on the following theorem:

39.2.2 Theorem (Lagrange multiplier theorem):

Let \((x_0, y_0) \in \mathbb{R}^2\) and
\[
f, g : B_r(x_0, y_0) \to \mathbb{R}
\]
be such that the following holds:

(i) Both the partial derivatives of \(f\) and \(g\) exist in \(B_r(x_0, y_0)\) and are continuous at \((x_0, y_0)\).

(ii) \(g(x_0, y_0) = 0\) and \(\nabla g(x_0, y_0) \neq (0, 0)\).

(iii) The function \(f\) has a local extremum at \((x_0, y_0)\), when restricted to \(C\), the level curve
\[
C = \{(x, y) \in \mathbb{R}^2 \mid g(x, y) = 0\}.
\]

Then,
\[
\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)
\]
for some \(\lambda \in \mathbb{R}\).

PROOF

39.2.2 Theorem (Lagrange multiplier theorem):

Let \((x_0, y_0) \in \mathbb{R}^2\) and
\[
f, g : B_r(x_0, y_0) \to \mathbb{R}
\]
be such that the following holds:

(i) Both the partial derivatives of \(f\) and \(g\) exist in \(B_r(x_0, y_0)\) and are continuous at \((x_0, y_0)\).

(ii) \(g(x_0, y_0) = 0\) and \(\nabla g(x_0, y_0) \neq (0, 0)\).

(iii) The function \(f\) has a local extremum at \((x_0, y_0)\), when restricted to \(C\), the level curve
\[
C = \{(x, y) \in \mathbb{R}^2 \mid g(x, y) = 0\}.
\]
Then, \( \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \) for some \( \lambda \in \mathbb{R} \).

**Proof**

Since \( \nabla g(x_0, y_0) \neq (0,0) \), we have

\[ g_x(x_0, y_0) \neq 0 \text{ or } g_y(x_0, y_0) \neq 0. \]

Suppose, \( g_y(x_0, y_0) \neq 0 \). Then, using implicit function theorem, we can find a function

\[ y : [x_0 - \delta, x_0 + \delta] \to \mathbb{R} \text{ such that } \]

\[ g(x, y(x)) = 0 \text{ for all } x \in [x_0 - \delta, x_0 + \delta]. \]

Hence, by chain rule,

\[ g_x(x_0, y_0) + g_y(x_0, y_0) y'(x_0) = 0. \]

Also, since \( f \) has a local extremum at the point \( (x_0, y_0) \) when restricted to \( C \), if we define

\[ \phi : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}, \quad \phi(x) = f(x, y(x)), \]

then \( \phi \) has a local extremum at \( x_0 \). Therefore,

\[ \phi'(x_0) = f_x(x_0, y_0) + f_y(x_0, y_0) y'(x_0) = 0. \]

It follows from the equations (32) and (33),

\[ f_y(x_0, y_0) g_x(x_0, y_0) = f_x(x_0, y_0) g_y(x_0, y_0), \]

and hence

\[ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0), \]

where

\[ \lambda = f_y(x_0, y_0)/g_y(x_0, y_0). \]

**39.2.3 Note (Lagrange's multiplier method):**

In view of the above theorem, to determine the absolute maximum/minimum of a function \( f(x, y) \) subject to the constraint \( g(x, y) = 0 \), we follow the following steps:

Step (i): Solve the equations

\[ \nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0. \]

\[ \text{---------(34)} \]

Let

\[ S_1 := \{(x, y) \mid \text{equation (34) is satisfied}\}. \]

Step (ii): Let

\[ S_2 := \{(x, y) \in S \mid g(x, y) = 0 \text{ or } f_y(x, y) \text{ does not exist, or } \nabla g(x_0, y_0) = (0,0)\}. \]

Step (iii): Evaluate \( f \) at each of the points in \( S_1 \cup S_2 \). Find \( M \), the largest of these values and \( m \), the smallest of these values.

Step (iv): Ensure that \( M \) and \( m \) are the required absolute maximum and absolute minimum of \( f \) respectively for the given constraints.

**39.2.4 Examples:**

(i) Let us find the maximum and the minimum of \( f(x, y) = xy \) on the unit circle, that is, subject the the constraint

\[ g(x, y) = x^2 + y^2 - 1 = 0. \]

Since the conditions of the theorem 39.2.1 are satisfied, we consider the equations

\[ \nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0, \]

\[ \text{---------(35)} \]
i.e.,
\[ y = 2 \lambda x, \quad x = 2 \lambda y \quad \text{and} \quad x^2 + y^2 - 1 = 0. \]

It is easy to check that the points \((x, y)\) that satisfy these equations are \((\pm 1/\sqrt{2}, \pm 1/\sqrt{2})\).

Since the unit circle is closed and bounded and \(f\) is continuous on it, \(f\) attains its absolute maximum/minimum on it, and are the largest/smallest of the values

\[ f(1/\sqrt{2}, 1/\sqrt{2}) = f(-1/\sqrt{2}, -1/\sqrt{2}) = \frac{1}{2} \]

and

\[ f(1/\sqrt{2}, -1/\sqrt{2}) = f(-1/\sqrt{2}, 1/\sqrt{2}) = -\frac{1}{2}. \]

Thus, \(f\) has absolute maximum \(1/2\), absolute minimum \(-1/2\).

(ii) Let us find the minimum of the function

\[ f(x, y) = x^2 + y^2 \quad \text{subject to the constraint} \quad g(x, y) = (x - 1)^3 - y^2 = 0. \]

The equations

\[ \nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0, \]

we have

\[ 2x = 3\lambda (x - 1)^2, \quad 2y = -2\lambda y, (x - 1)^3 - y^2 = 0. \]

have no solutions for \(y \neq 0\). For \(y = 0, g(x, y) = 0\), gives have \(x = 1\). But then

\[ \nabla g(1, 0) = 0 \quad \text{and} \quad \nabla f(1, 0) = (2, 0). \]

Thus, the equation

\[ \nabla f(1, 0) = \lambda \nabla g(1, 0) \]

is not satisfied for any \(\lambda \in \mathbb{R}\).

Hence, the condition \(\nabla g(x_0, y_0) \neq (0, 0)\) cannot be dropped in theorem 39.2.1. However, \(f(x, y)\) is the distance between origin and a point on the surface \((x - 1)^3 = y^2\). Geometrically it is obvious that the minimum of \(f\) is \(1\) and this is attained at \((1, 0)\).

### 39.2.5 Constrained extremum for three Variables:

There is a result analogous to the two variable, to solve the problem of constrained maxima / minima for functions of three variables. We solve the equations

\[ \nabla f(x, y, z) = \lambda \nabla g(x, y, z), g(x, y, z) = 0, \]

in the unknowns \(x, y, z\) and \(\lambda\) at which \(\nabla g(x, y, z) \neq (0, 0, 0)\) and compare the values of \(f\) at these points to locate the constrained maxima/minima of \(f\).
39.2.6 Examples:

Let us find the points on the surface \( z^2 = xy + 4 \) closest to the origin. This is same as minimizing the function

\[ f(x, y, z) = x^2 + y^2 + z^2 \]

with constraint \( g(x, y, z) = xy + 4 - z^2 \).

Note that although the set

\[ S = \{(x, y, z) \in \mathbb{R}^3 \mid g(x, y, z) = 0\} \]

is not bounded, the set

\[ S_r = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq r\} \]

is closed and bounded, where \( r = \sqrt{x_0^2 + y_0^2 + z_0^2} \) for some \( (x_0, y_0, z_0) \in \mathbb{R}^3 \). Further, the minimum of \( f \) on \( S_r \) equals the minimum of \( f \) on \( S \), which exists as \( f \) is continuous. To find this, we solve the equations

\[ \nabla f = \lambda \nabla g, \quad \text{i.e.,} \quad 2x = \lambda y, \quad 2y = \lambda x, \quad 2z = -2\lambda z. \]

Since \( \lambda = 0 \) implies \( (x, y, z) = (0, 0, 0) \) and \( g(0, 0, 0) \neq 0 \),

we may assume that \( \lambda \neq 0 \). Then, it is easy to see that the only common solutions of

\[ \nabla f = \lambda \nabla g \quad \text{and} \quad g = 0 \]

are the points

\( (0,0,2), (0,0,-2), (2,-2,0) \) and \( (-2,2,0) \).

Further,

\[ f(0,0,\pm 2) = 4 \quad \text{and} \quad f(\pm 2,\mp 2,0) = 8. \]

Thus, \( (0,0,\pm 2) \) are the points on the surface \( z^2 = xy + 4 \) closest to the origin.

39.2.7 Remark:

The method of Lagrange's multipliers extents when we have more than one constraint. Suppose we want to find extremum a function \( f(x, y, z) \) with constraints

\[ g(x, y, z) = 0 \quad \text{and} \quad h(x, y, z) = 0, \]

where \( g \) and \( h \) have continuous partial derivatives in a neighborhood of \( (x_0, y_0, z_0) \). These can be found by comparing the values of \( f \) at points which satisfy the simultaneous equations
\[ \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z), \]
\[ g(x,y,z) = 0 = h(x,y,z), \]
and for which
\[ \nabla g(x,y,z) \neq (0,0,0), \nabla h(x,y,z) \neq (0,0,0) \]
and
\[ \nabla g(x,y,z) \text{ is not parallel to } \nabla h(x,y,z). \]

**Example:**

Let us analyze the problem of finding the points on the intersection of the planes
\[ x + y + z = 1 \text{ and } 3x + 2y + z = 6 \]
that are closest to the origin. This is same as finding the minimum value of
\[ f(x,y,z) = x^2 + y^2 + z^2, \]
with constraints,
\[ g(x,y,z) = x + y + z - 1 = 0 \text{ and } h(x,y,z) = 3x + 2y + z - 6 = 0. \]
The equations to be solved are
\[ x = \frac{\lambda + 3\mu}{2}, \quad y = \frac{\lambda + 2\mu}{2}, \quad z = \frac{\lambda + \mu}{2}, \]
\[ x + y + z - 1 = 0, \quad 3x + 2y + z - 6 = 0 \]
Substituting the values of \( x, y, z \) from the first three in the last two equations gives
\[ 3\lambda + 6\mu = 2 \quad \text{and} \quad 3\lambda + 7\mu = 6. \]
This gives
\[ \mu_0 = 4 \text{ and } \lambda_0 = -22/3, \text{ and hence } (x_0, y_0, z_0) = (7/3, 1/3, -5/3). \]
That this is the required point, can be justified as in the previous example.

**Practice Exercises**

1. The temperature at a point \( (x,y,z) \) in 3-space is given by \( T(x,y,z) = 400xyz^2 \). Find the highest temperature on the unit sphere \( x^2 + y^2 + z^2 = 1 \).
   **Answer**

2. Find the point nearest to the origin on the surface defined by the equation \( z = xy + 1 \).
   **Answer**

3. Using the Lagrange's method of multiplier, show that the minimum value of \( f \) is 0 and is attained at
and the maximum value of $f(x, y, z) = x^2y^2z^2$ subject to the constraint that $(x, y, z)$ lies on the unit sphere is $\frac{1}{27}$ and it is attained at $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Using these, deduce the A.M.-G.M. inequality: for three nonnegative real numbers $x, y, z \in \mathbb{R}$, 

$$\left( x^2y^2z^2 \right)^{\frac{1}{3}} \leq \frac{(x^2 + y^2 + z^2)}{3}.$$ 

(4) A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point $(x, y, z)$ on the surface of the probe is given by

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$ 

Find the hottest point on the surface of the probe.

**Answer**

(5) Maximize the function

$$f(x, y, z) = xyz$$

subject to the constraints

$$x + y + z = 40 \text{ and } x + y = z.$$ 

**Answer**

(6) Minimize the quantity

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints

$$x + 2y + 3z = 6 \text{ and } x + 3y + 9z = 9.$$ 

**Answer**

**Recap**

In this section you have learnt the following:

- The notion of constrained maxima/minima for functions of several variables.

- Lagrange's Method of finding constrained maxima/minima.