Module 12: Total differential, Tangent planes and normals

Lecture 36: Mean value theorem and Linearization [Section 36.1]

Objectives
In this section you will learn the following:

- Mean value theorem for functions of several variables
- Linear approximations for functions of several variables

36.1 Mean value theorem and Linearization

36.1.1 Definition:

Let $D \subset \mathbb{R}^2$ and $P(x_0, y_0), Q(x_1, y_1) \in D$. Let

$h := x_1 - x_0$, and $k := y_1 - y_0$.

Then,

$L(P, Q) := ((x_0 + th, y_0 + tk) | 0 < t < 1)$

and

$L(P, Q) := ((x_0 + th, y_0 + tk) | 0 \leq t \leq 1)$

are the open and the closed straight line segments joining $(x_0, y_0)$ and $(x_1, y_1)$. 
36.1.2 Mean Value Theorem:

Let $D \subset \mathbb{R}^2$ and $P(x_0, y_0), Q(x_1, y_1) \in D$ be such that every point of $L(P, Q)$ is an interior point of $D$. Let $f : D \to \mathbb{R}$ be such that the following hold:

(i) Both $f_x$ and $f_y$ exist at every interior point of $D$, and are continuous at every point of $L(P, Q)$.

(ii) $f$ is continuous at both $P(x_0, y_0)$ and $Q(x_1, y_1)$.

Then there exists some $\theta \in (0, 1)$ such that

$$f(x_1, y_1) - f(x_0, y_0) = h \cdot f_x(x_0 + \theta h, y_0 + \theta k) + k \cdot f_y(x_0 + \theta h, y_0 + \theta k).$$

Proof

Define $\phi : [0, 1] \to \mathbb{R}$ by

$$\phi(t) = f(x_0 + th, y_0 + tk), t \in [0, 1].$$

Then, $\phi$ is continuous on $[0, 1]$. Also, by the chain rule, $\phi$ is differentiable at every $t \in (0, 1)$ with

$$\phi'(t) = hf_x(x_0 + th, y_0 + tk) + kf_y(x_0 + th, y_0 + tk).$$

Hence by the mean value theorem for functions of one variable, there exists some $\theta \in (0, 1)$ such that

$$\phi(1) - \phi(0) = \phi'(\theta).$$

Thus,

$$f(x_1, y_1) - f(x_0, y_0) = h \cdot f_x(x_0 + \theta h, y_0 + \theta k) + k \cdot f_y(x_0 + \theta h, y_0 + \theta k).$$
36.1.3 Remark:

(i) The mean value theorem for functions of two variables has consequences analogous to those of the mean value theorem for functions of one variable. For example, if every point of $D$ is an interior point of $D$ and any two points of $D$ can be joined by a finite number of straight line segments which lie in $D$, then any function $f : D \to \mathbb{R}$ such that $f_x = f_y = 0$ at every point of $D$, $f$ must be a constant function.

(ii) As in the case of functions of a single variable, Taylor's theorem holds for functions of several variables also. The interested reader may consult an advanced book on calculus of several variables.

36.1.4 Definition:

Let $(x_0, y_0) \in \mathbb{R}^2$ and $f : B_r(x_0, y_0) \to \mathbb{R}$ be such that $f_x, f_y$ exist on $B_r(x_0, y_0)$ and are continuous at $(x_0, y_0)$. Then,

$$T(x, y) := f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0)$$

is a linear function called the linear (or tangent plane) approximation of $f$ for $(x, y)$ near the point $(x_0, y_0)$.

36.1.5 Note (Error estimate):

(i) If

$$h = x - x_0 \text{ and } k = y - y_0,$$

then using theorem 36.1.2, we can find $\theta \in (0, 1)$ such that at points $(x, y)$ close to $(x_0, y_0)$,

$$f(x, y) - f(x_0, y_0) = h f_x(x_0 + \theta h, y_0 + \theta k) + k f_y(x_0 + \theta h, y_0 + \theta k).$$

Thus, the error in using linear approximation is given by

$$e_1(x, y) := f(x, y) - T(x, y)$$

$$= f(x, y) - f(x_0, y_0) - h f_x(x_0, y_0) - k f_y(x_0, y_0)$$

$$= h f_x(x_0 + \theta h, y_0 + \theta k) + k f_y(x_0 + \theta h, y_0 + \theta k) - h f_x(x_0, y_0) - k f_y(x_0, y_0)$$

$$= h f_x(x_0 + \theta h, y_0 + \theta k) f_x(x_0, y_0) + k[f_y(x_0 + \theta h, y_0 + \theta k) - f_y(x_0, y_0)].$$

It follows from the continuity of the partial derivatives that the error $e_1(x, y)$ tends to zero as $(x, y) \to (x_0, y_0)$, i.e., as $(h, k)$ tends to zero. In fact, it tends to zero ‘faster’ than as $(h, k)$ tends to zero, since

$$\left| \frac{e_1(x, y)}{\sqrt{h^2 + k^2}} \right| \leq |f_x(x_0 + \theta h, y_0 + \theta k) - f_x(x_0, y_0)| + |f_y(x_0 + \theta h, y_0 + \theta k) - f_y(x_0, y_0)|$$

$$\to 0 \text{ as } (h, k) \to 0.$$

(ii) If the second order partial derivatives of $f$ exist and are continuous in some open ball $B_r(x_0, y_0)$ centered at $Q(x_0, y_0)$, then for $Q(x, y) \in B_r(x_0, y_0)$, it can be shown that the error in linear approximation can be estimated as follows:
where \( M_2(x, y) \) is a real number such that

\[
M_2(x, y) \geq \sup \{ |f_{xx}(a, b)|, |f_{yy}(a, b)|, |f_{xy}(a, b)| (a, b) \in \mathcal{L}(F, Q) \}.
\]

### 36.1.6 Example:

Let

\[
f(x, y) = \frac{1}{1 - x - y} \text{ for } (x, y) \in \mathbb{R}^2 \text{ with } x + y \neq 1.
\]

Then,

\[
f_x(0, 0) = 1 = f_y(0, 0).
\]

Thus, the linear approximation to \( f \) for \( (x, y) \) near \((0, 0)\) is

\[
T(x, y) = 1 + x + y.
\]

To estimate the error, we note that

\[
f_{xx}(a, b) = f_{yy}(a, b) = f_{xy}(a, b) = \frac{2}{(1 - a - b)^3}.
\]

Thus, we can take

\[
M_2(x, y) = \frac{2}{(1 - x - y)^3} \text{ if } 0 < x + y < 1
\]

and

\[
M_2(x, y) = 2 \text{ if } x + y \leq 0.
\]

For example, if both \(|x|, |y| < 0.1\), then

\[
|e_1(x, y)| \leq \left( \frac{2}{(1 - 0.1 - 1)^3} \right) \left( \frac{(0 - 0.1) + (0 - 0.1))^2}{2} \right) = 0.7813 \text{ if } x + y > 0
\]

and

\[
|e_1(x, y)| \leq \left( \frac{2}{2} \right) (0 - 0.1) + (0 - 0.1))^2 = 0.04 \text{ if } x + y \leq 0.
\]

### 36.1.7 Example:

Consider a rectangular box of length \( x = 50 \text{ cms} \), width \( y = 20 \text{ cms} \) and height \( z = 15 \text{ cms} \). We want to find the percentage error made in measuring the volume if an error of \( \pm 0.1 \text{ mm} \) is made in measuring each dimension of the box. Since the volume \( V \) is given by
\[ V(x, y, z) = xyz, \]

with

\[ (x_0, y_0, z_0) = (50, 20, 15) \text{ and } (x_1, y_1, z_1) = (x_0 \pm 0.01, y_0 \pm 0.01, z_0 \pm 0.01), \]

the error is given by

\[
\varepsilon_1(x, y, z) = (\pm 0.01) \left[ f_x(50, 20, 15) + f_y(50, 20, 15) + f_z(50, 20, 15) \right]
\]

\[
= (\pm 0.01) \left[ (20)(15) + (50)(15) + (50)(20) \right]
\]

\[
= (\pm 0.01)(20\, 50)
\]

\[
= \pm 20.5 \text{cm}^3
\]

The actual volume is

\[ V(50, 20, 15) = 15,000 \text{cm}^3. \]

Thus, the percentage error is

\[
\frac{\varepsilon_1(x, y, z)}{V(x, y, z)} \times 100 = \frac{20.5}{15,000} \times 100 = 0.14\%
\]

**Practice Exercises**

(1) For the following functions, find the linear approximation at the a bound for the error is approximation in the specified region:

(i) \( f(x, y) = 2x^2 - 2xy + y^2 + 6, P = (3, 2), \) and

\[ |x-3|<0.1, |y-2|<0.1. \]

(ii) \( f(x, y, z) = xy + 2yz - 3xz, P = (1, 1, 0), \) and

\[ |x-1|<0.01, |y-1|<0.01, |z-1|<0.01. \]

**Answer**

(2) Let \( f(x, y) = x^2 + y^2. \) Find the linear approximation to \( f \) at \( P = (0, 0). \) Further find the points \((x, y) \neq (0, 0)\) such that the error \( \varepsilon_1(x, y) \) has the property

\[
\left| \frac{\varepsilon_1(x, y)}{\sqrt{x^2 + y^2}} \right| < 10^{-3}.
\]
Recap
In this section you have learnt the following

- Mean value theorem for functions of several variables
- Linear approximations for functions of several variables