34.1 Gradient of a scalar field

We have seen that for a function \( f(x, y, z) \) the partial derivatives \( f_x, f_y, f_z \), whenever they exist, play an important role. This motivates the following definition.

34.1.1 Definition:

Let \( f(x, y, z) \) and \( (x_0, y_0, z_0) \) be differentiable at \( (x_0, y_0, z_0) \). If each of \( f_x, f_y \) and \( f_z \) exist at a point \( (x_0, y_0, z_0) \), then the vector \( (f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)) \) is called the gradient vector of \( f \) at \( (x_0, y_0, z_0) \), and is denoted by

\[
(\nabla f)(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)).
\]

For a function of 2-variables, it is given by

\[
(\nabla f)(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0, z_0)).
\]

34.1.2 Theorem:

Let \( (x_0, y_0, z_0) \) be differentiable at \( (x_0, y_0, z_0) \).

(i) For every unit vector \( u \in \mathbb{R}^3 \), \( (D_u)(x_0, y_0, z_0) \) exist and

\[
(D_u f)(x_0, y_0, z_0) = (\nabla f)(x_0, y_0, z_0) \cdot u.
\]
(ii) Suppose \( D \) is such that any two points in it can be joined by line segments parallel to axes and \((\nabla f)(x,y,z) = 0\) for all \((x,y,z) \in D\), then \( f \) is constant in \( D \).

**Proof**

**34.1.2 Theorem:**

Let \((x_0, y_0, z_0) \in D \subseteq \mathbb{R}^3\) and \( f : D \to \mathbb{R} \) be differentiable at \((x_0, y_0, z_0)\).

(i) For every unit vector \( u \in \mathbb{R}^3 \), \((D_u f)(x_0, y_0, z_0)\) exist and

\[
(D_u f)(x_0, y_0, z_0) = (\nabla f)(x_0, y_0, z_0) \cdot u.
\]

(ii) Suppose \( D \) is such that any two points in it can be joined by line segments parallel to axes and \((\nabla f)(x,y,z) = 0\) for all \((x,y,z) \in D\), then \( f \) is constant in \( D \).

Proof

The proof of (i) follows from theorem 33.2.4. To prove (ii) first note that the given condition \((\nabla f)(x,y,z) = 0\) for all \((x,y,z) \in D\), implies that

each of \( f_x, f_y, f_z = 0 \) in \( D \).

Let \( A, B \in D \) be such that \( A \) and \( B \) can be joined by a path as shown in figure below, where \( AC, BC \) are parallel to axes.

![Figure 1](image)

Then, by one variable case,

\( f(A) = f(C) = f(B) \).

Thus, if any two points in \( D \) can be joined by a piecewise linear path, moving parallel to axes only, then

\((\nabla f)(x,y) = 0\) for all \( x,y \in D \) implies that \( f \) is constant in \( D \).
34.1.3 Example:

Let

\[ f(x, y, z) = x^2 y - yz^3 + z. \]

Then

\[
\begin{align*}
  f_x (x, y, z) &= 2xy, \\
  f_y (x, y, z) &= x^2 - z^3, \\
  f_z (x, y, z) &= -3yz^2 + 1.
\end{align*}
\]

Obviously, each of \( f_x, f_y, f_z \) is a continuous function everywhere. Then, \( f \) is differentiable and for every unit vector \( u \)

\[
(D_u f)(x_0, y_0, z_0) = (\nabla f)(x_0, y_0, z_0) \cdot u.
\]

For example, if we want to find the directional derivative of \( f \) at the point \((1, -2, 0)\), in the direction of the vector \( \mathbf{v} \), than we take

\[
\mathbf{u} = \frac{2\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{\sqrt{a}} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}
\]

and

\[
f_x (1, -2, 0) = -4, \quad f_y (1, -2, 0) = 1, \quad f_z (1, -2, 0) = 1.
\]

Thus

\[
(D_u f)(1, -1, 0) = (-4, 1, 1) \cdot \left( \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) = -3.
\]

34.1.4 Remark:

The formula

\[
(D_u f)(x_0, y_0, z_0) = (\nabla f)(x_0, y_0, z_0) \cdot u
\]

may not hold if \( f_x, f_y \) or either of \( f_y \) is discontinuous at \((x_0, y_0, z_0)\).

For example, consider \( f: \mathbb{R}^2 \to \mathbb{R} \) given by

\[
f(0, 0) = 0 \quad \text{and} \quad f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{for} \quad (x, y) \neq (0, 0).
\]

We have

\[
\nabla f(0, 0) = (1, 0),
\]

and for any unit vector \( u = (u_1, u_2) \),

\[
(D_u f)(0, 0) = u_1^2.
\]

Thus,

\[
(D_u f)(0, 0) \neq (\nabla f(0, 0)) \cdot u, \quad \text{whenever} \quad u_1 \neq 0, 1, -1.
\]
Note that for \((x_0, y_0) \neq (0, 0)\), we have
\[
f_x(x_0, y_0) = \frac{x_0^4 + 3x_0^2y_0^2}{(x_0^2 + y_0^2)^2} \quad \text{and} \quad f_y(x_0, y_0) = \frac{-2x_0^3y_0}{(x_0^2 + y_0^2)^2}.
\]

It is easy to see that both \(f_x\) and \(f_y\) are discontinuous at \((0, 0)\).

We describe next some geometric properties of the gradient.

34.1.5 Theorem:

Let \(f : D \subseteq \mathbb{R}^3 \to \mathbb{R}\) be differentiable at \((x_0, y_0, z_0) \in D\) so that that
\[
\nabla f(x_0, y_0, z_0) \neq (0, 0, 0).
\]

Let \(u = (u_1, u_2, u_3)\) be a unit vector. Then the following holds:

(i) Near the point \((x_0, y_0, z_0)\), the direction in which \(f\) increases most rapidly is that of \(\nabla f(x_0, y_0, z_0)\).

(ii) Near the point \((x_0, y_0, z_0)\), the direction in which \(f\) decreases most rapidly is the one opposite to that of \(\nabla f(x_0, y_0, z_0)\).

(iii) Near the point \((x_0, y_0, z_0)\), the directions perpendicular to that of \(\nabla f(x_0, y_0)\) are the directions of no change in \(f\).

Proof

By definition, we have
\[
(\nabla f \cdot u)(x_0, y_0, z_0) = (\nabla f(x_0, y_0, z_0)) \cdot u = |\nabla f(x_0, y_0, z_0)| \cos \theta,
\]
where \(\theta \in [0, \pi]\) is the angle between \(\nabla f(x_0, y_0, z_0)\) and \(u\). Since \(-1 \leq \cos \theta \leq 1\), we have
\[
(\nabla f \cdot u)(x_0, y_0, z_0) \text{ is maximum when } \cos \theta = 1, \theta = 0.
\]
Thus, near \((x_0, y_0, z_0)\),
\[
u = \frac{\nabla f(x_0, y_0, z_0)}{|\nabla f(x_0, y_0, z_0)|}
\]
is the direction in which \(f\) increases most rapidly.

The value of \(\nabla f \cdot u(x_0, y_0, z_0)\) is minimum when \(\cos \theta = -1\), that is, when \(\theta = \pi\). Thus, near
Finally, \( (D_{x_0}f)(x_0, y_0, z_0) = 0 \) when \( \cos \theta = 0 \), that is, when \( \varphi = \pi/2 \). Thus, near \((x_0, y_0, z_0)\),
\[
\mathbf{u} = \pm \frac{f_y(x_0, y_0, z_0) \mathbf{i} - f_x(x_0, y_0, z_0) \mathbf{j}}{|f(x_0, y_0, z_0)|}
\]
are the directions of no change in \( f \).

### 34.1.6 Note:
In case \( \nabla f(x_0, y_0, z_0) = (0, 0, 0) \), we have \( (D_{x_0}f)(x_0, y_0, z_0) = 0 \) for every \( u \), and hence near \((x_0, y_0, z_0)\), \( f \) has no rate of change in all directions.

### 34.1.7 Example:
Consider \( f : \mathbb{R}^2 \to \mathbb{R} \) given by Suppose
\[
f(x, y) = 4 - x^2 - y^2 \text{ for } (x, y) \in \mathbb{R}^2.
\]
We have
\[
f_x = -2x, f_y = -2y.
\]
At \((x_0, y_0) = (1, 1)\)
\[
\nabla f(1, 1) = (-2, -2).
\]
Thus, on the surface \( z = f(x, y) \) near \((1, 1)\),
\[
\frac{\nabla f(1, 1)}{|\nabla f(1, 1)|} = \frac{(-2, -2)}{2\sqrt{2}} = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)
\]
is the direction of steepest ascent.

while in the reverse direction, namely,
\[
\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)
\]
is direction of steepest descent.

The directions of no change are
\[
\pm \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right).
\]
Since \( \nabla f(0, 0) = (0, 0) \), rate of change of \( f \) is zero in every direction at \((0, 0)\).

### 34.1.8 Example:
Let
\[
f(x, y) = 20 - 4x^2 - y^2
\]
represent the temperature of a metallic sheet. Starting at the point \((2, 1)\) let us find the continuous path
\[
\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j},
\]
that will give the direction of maximum increase in temperature. Since, the direction to this path at any time point \( t \) is
\[
\mathbf{r}'(t) = x'(t) \mathbf{i} + y'(t) \mathbf{j},
\]
and that has to be of maximum increase of \( f \), we should have
\[ \alpha r'(t) = \nabla f, \text{ for some scalar } \alpha. \]

That is,
\[ \alpha x'(t) \mathbf{i} + \alpha y'(t) \mathbf{j} = -8x \mathbf{i} - 2y \mathbf{j}, \]
i.e.,
\[ \alpha x'(t) = -8x, \; \alpha y'(t) = -2y. \]
This gives us the differential equation
\[ \frac{dy}{dx} = \frac{2y}{8x} = \frac{y}{4x}. \]
A solution to which is
\[ x = ky^4, \; k \text{ some scalar.} \]
Since, this passes through \((1,2)\), we have
\[ 2 = k. \]
Thus, the required path is \( x = 2y^4 \).

**Practice Exercises**

(1) Find the gradient for the following functions at the indicated point \( P \) and its directional derivative at \( P \) in the direction of the indicated point \( Q \):

(i) \( f(x, y) = \sqrt{xy} e^y, \; P = (1, 1), \; Q = (0, -1) \).

(ii) \( f(x, y, z) = x^3y^2z^5 - 2xz + yz + 3x, \; P = (-1, -2, 1), \; Q = (0, 0, -1) \).

**Answers**

(2) For the following functions, find the direction of maximum increase at the indicated point:

(i) \( f(x, y, z) = \sin xy + \cos yz, \; P = (-3, 0, 7) \).

(ii) \( f(x, y, z) = 2xyz + y^2 + z^2, \; P = (2, 1, 1) \).

**Answers**

(3) The temperature at a point \((x, y, z)\) on the surface of a body is given by

\[ T(x, y, z) = 2x^2 - y^2 + 4z^2. \]

Find the rate of change of temperature at the point \( P = (1, -2, 1) \) in the direction of the vector \( 4\mathbf{i} - \mathbf{j} + 2\mathbf{k} \).

In what direction at \( P \), the temperature is decreasing most rapidly?

**Answers**

(4) If \( z = f(x, y) \) is a differentiable function, where \( x = x(t) \) and \( y = y(t) \) are also differentiable with respect to \( t \), compute \( \frac{dz}{dt} \) in terms of \( \nabla z \).

**Answers**

(5) Let \( f(x, y) \) be a differentiable function such that
for any two fixed vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^2 \) such that \( \mathbf{u} \neq \alpha \mathbf{v} \) for any constant \( \alpha \). Show that \( (D_{\mathbf{w}}f)(x, y) = 0 \) for all \( \mathbf{w} \in \mathbb{R}^2 \).

(6) Let \( f(x, y) \) be such that

(i) \( f_x(x, y) \) and \( f_y(x, y) \) exist for all \( (x, y) \in B_r(1, 2) \) for some \( r > 0 \) and are continuous at \( (1, 2) \).

(ii) The directional derivative of \( f \) at \( (1, 2) \) in the direction toward \( (2, 3) \) is \( 2\sqrt{2} \).

(iii) The directional derivative of \( f \) at \( (1, 2) \) in the direction toward \( (1, 0) \) is \(-3\). Find \( f_x(1, 2), f_y(1, 2) \) and the directional derivative of \( f \) at \( (1, 2) \) in the direction toward \( (4, 6) \).

**Answers**

(7) Let \( f : D \subseteq \mathbb{R}^3 \to \mathbb{R} \) be such that all \( f_x, f_y, f_z, g_x, g_y \) and \( g_z \) exist in \( B_r((x_0, y_0)) \), for some \( r > 0 \).

Prove the following:

(i) \( (\nabla f)(f \pm g) = (\nabla f) \pm (\nabla g) \).

(ii) \( \nabla (fg) = f(\nabla g) + (\nabla f)g \).

(iii) \( \nabla (\alpha f) = \alpha (\nabla f) \), for every \( \alpha \in \mathbb{R} \).

**Recap**

In this section you have learnt the following

- The notions gradient vector

- The relation of gradient with the directional derivative