Module 6: Definition of Integral

Lecture 17: Fundamental theorem of calculus [Section 17.1]

Objectives
In this section you will learn the following:

- Fundamental theorem of calculus, which relates integration with differentiation.

17.1 Fundamental Theorem of Calculus

In this lecture, we describe an important theorem which connects integration with differentiation. We first make a simple observation:

17.1.1 Proposition:

Let \( f : [a, b] \to \mathbb{R} \) be an integral function. If \( A \in \mathbb{R} \) is such that
\[
L(P, f) \leq A \leq U(P, f)
\]
for every partition \( P \) of \( [a, b] \), then \( f \) is integrable and
\[
A = \int_a^b f(x)\,dx.
\]

Proof:

17.1.1 Proposition:

Let \( f : [a, b] \to \mathbb{R} \) be an integral function. If \( A \in \mathbb{R} \) is such that
\[
L(P, f) \leq A \leq U(P, f)
\]
for every partition \( P \) of \( [a, b] \), then \( f \) is integrable and
\[
A = \int_a^b f(x)\,dx.
\]

Proof:
Since \( f \) is integrable, there exists a sequence \( \left( P_n \right)_{n \in \mathbb{N}} \) of refinement partitions of \([a, b]\) such that
\[
\lim_{n \to \infty} \left[ U(P_n, f) - L(P_n, f) \right] = 0.
\]
By the given hypothesis,
\[
L(P_n, f) \leq A \leq U(P_n, f),
\]
for every \( n \geq 1 \).
Hence,
\[
\lim_{n \to \infty} U(P_n, f) = A = \lim_{n \to \infty} L(P_n, f).
\]
Thus, by definition
\[
A = \int_a^b f(x) \, dx.
\]

**17.1.2 Fundamental Theorem of Calculus - I (FTC-I):**

Let \( f, F : [a, b] \to \mathbb{R} \) be functions with the following properties:

(i) \( f \) is integrable on \([a, b]\).

(ii) \( F \) is continuous on \([a, b]\).

(iii) \( F \) is differentiable on \((a, b)\) with \( F'(x) = f(x) \) for all \( x \in (a, b) \).

Then,
\[
\int_a^b f(t) \, dt = F(b) - F(a).
\]

**Proof:**

Let \( P = \{a = x_0, x_1, \ldots, x_n = b\} \) be any partition of \([a, b]\). Then
\[
F(b) - F(a) = F(x_n) - F(x_0) = \sum_{k=1}^{n} \left[ F(x_k) - F(x_{k-1}) \right].
\]
By the mean value theorem for \( F \) on \([x_{k-1}, x_k]\), there exists \( c_k \in (x_{k-1}, x_k) \) such that
\[
F(x_k) - F(x_{k-1}) = F'(c_k)(x_k - x_{k-1}).
\]
Since \( F'(x) = f(x) \), for all \( x \in [a, b] \), we have
\[
F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1}).
\]
From equations (2) and (3), we get
\[
F(b) - F(a) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}).
\]
Thus, for every partition \( P \) of \([a, b]\),
\[
L(P, f) \leq F(b) - F(a) \leq U(P, f).
\]
Hence, \( F(b) - F(a) = \int_a^b f(x) \, dx \).

17.1.3 Examples:

(i) Since, for every \( n \geq 1 \),
\[
\frac{d}{dx} (x^n) = nx^{n-1},
\]
for every interval \([a, b]\),
\[
\int_a^b nx^{n-1} \, dx = b^n - a^n,
\]
i.e.,
\[
\int_a^b x^{n-1} \, dx = \frac{b^n - a^n}{n}.
\]

(ii) Since
\[
\frac{d}{dx} (\sin x) = \cos x,
\]
for \( a, b \in \mathbb{R} \) with \( a < b \),
\[
\int_a^b \cos x \, dx = \sin(b) - \sin(a).
\]

(iii) For the function \( f(x) = \exp(x), x \in \mathbb{R} \)
\[
\frac{d}{dx} (\exp(x)) = \exp(x).
\]
Hence, for \( a, b \in \mathbb{R} \) with \( a < b \),
\[
\int_a^b e^x \, dx = e^b - e^a.
\]

17.1.4 Definition :

Let \( f, F : [a, b] \to \mathbb{R} \) be functions such that \( F \) is differentiable and
\[
F'(x) = f(x) \text{ for all } x \in (a, b).
\]
Then, \( F \) is called an antiderivative of \( f \) on \([a, b]\).

17.1.5 Examples :

(i) Let \( F(x) = x^2, x \in \mathbb{R} \). Since \( f'(x) = 2x \), which is a continuous function, an antiderivative of \( f(x) = 2x \) is
\[
F(x) = x^2.
\]
Infact, for any \( n \neq -1 \), since
\[ F(x) = x^2 \]
\[ \frac{d}{dx} (x^n) = nx^{n-1}, \]
we deduce that the function \( f(x) = x^n \) has antiderivative
\[ F(x) = \frac{x^{n+1}}{n+1}, \quad x \in \mathbb{R}, \; n \neq -1 \]

(ii) For \( F(x) = \cos x, \; x \in \mathbb{R}, \; F'(x) = -\sin x \), implies that \( f(x) = \sin x \) has an antiderivative, namely
\[ f(x) = \cos x. \]

17.1.6 Remark:
If \( F(x) \) is an antiderivative of \( f(x) \), then clearly
\[ G(x) := F(x) + c, \; c \text{ a fixed constant}, \]
is also an antiderivative of \( f \). Thus antiderivative of a function \( f(x) \) is not unique. Any two antiderivatives differ by a constant.

17.1.7 Definition:
Let \( f : [a, b] \to \mathbb{R} \). The set of all the antiderivatives of \( f \) is denoted by
\[ \int f(x) \, dx, \]
and is called the indefinite integral or just integral of \( f \). Since any of two elements of this set differ only by a constant, we also write
\[ \int f(x) \, dx = F(x) + c, \]
where \( F \) is some antiderivative of \( f \).

17.1.8 Examples:
In view of examples 17.2.5, we can write
\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \; n \neq -1 \]
and
\[ \int \cos x \, dx = \sin x + c. \]

In view of theorem 17.1.1, since the knowledge about the antiderivative of a function is useful in calculating the integral of the function, it is natural to ask the question:
Given a function \( f : [a, b] \to \mathbb{R} \), can we always find an antiderivative of \( f \) ?
The answer to this is given by the following:

17.1.9 Fundamental Theorem of Calculus - II (FTC - II):
Let \( f : [a, b] \to \mathbb{R} \) be continuous. Then
\[ F(x) = \int_a^x f(t) \, dt, \quad x \in [a, b], \]
is differentiable with \( \frac{d}{dx} F(x) = f(x) \), i.e., has an anti-derivative, namely \( F \).
\[ f(x) = F'(x) \quad f \]

17.1.10 Remark:

Though the above theorem tells us that every continuous function has an anti-derivative, it may not be always possible to find it explicitly. Some methods that help us to do this, are discussed in the next section.

**PRACTICE EXERCISES**

1. Let \( f \) have an antiderivative \( F \) and \( g \) have an antiderivative \( G \). Find an antiderivative of the following in terms of \( F \) and \( G \):
   
   - (i) \( \alpha f, \alpha \in \mathbb{R} \).
   - (ii) \( f + g \).

2. Show that if \( f : [a, b] \to \mathbb{R} \) is continuous and \( \beta \in \mathbb{R} \) is given, then there is a unique antiderivative \( F \) of \( f \) such that \( F(\alpha) = \beta \) for a given \( \alpha \in [a, b] \).

3. For the following \( f \), find unique antiderivative \( F \) with the specified values at a specified point:
   
   - (i) \( f(x) = 3x^2, \quad F(2) = 10 \).
   - (ii) \( f(x) = x^2 + x^3 + x^4, \quad F(1) = 0 \).
   - (iii) \( f(x) = x^{\frac{1}{3}}, \quad F(1) = 0 \).

4. Find the average values of the following functions over the indicated intervals:
   
   - (i) \( f(x)3x^2 - 2x, \quad [0, 2] \).
   - (ii) \( f(x) = 4 - x^2, \quad [-1, 1] \).
   - (iii) \( f(x) = \cos x, \quad [0, 3\pi/2] \).

**Recap**

In this section you have learnt the following:

- Fundamental theorem of calculus, which relates integration with differentiation.

(Section 17.2)

**Objectives**

In this section you will learn the following:

- Integration by parts formula
17.2 Applications of fundamental theorem of calculus

17.2.1 Theorem (Integration by Parts):

Let \( F, G : [a, b] \to \mathbb{R} \) be differentiable functions such that both \( F', G' \) are Riemann integrable on \([a, b]\). Then

\[
\int_a^b F(x)G'(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\,dx.
\]

**Proof:**

Note that, by the product rate for differentiation

\[
(FG)' = F'G + FG'.
\]

Since both \( F'G \) and \( FG' \) are integrable, by FTC-I, we have

\[
F(b)G(b) - F(a)G(a) = \int_a^b (FG)'(x)\,dx
= \int_a^b \left[F'(x)G(x) + F(x)G'(x)\right]dx
= \int_a^b F'(x)G(x)\,dx + \int_a^b F(x)G'(x)\,dx.
\]
17.2.2 Theorem (Integration by direct Substitution):

Let $f : [a, b] \to \mathbb{R}$, $g : [c, d] \to \mathbb{R}$ be functions such that

(i) $f$ is continuous on $[a, b]$.

(ii) $g$ is differentiable on $[c, d]$ with $g(c) = a$ and $g(d) = b$.

(iii) $g'$ Riemann integrable on $[c, d]$.

Then

$$
\int_c^d f(g(t))g'(t) \, dt = \int_a^b f(x) \, dx.
$$

**Proof:**

Since $f$ is continuous, by FTC-I, $f$ has an antiderivative, say $F$. Then $F'(x) = f(x)$ for all $x$.

Also by the chain rule,

$$(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Thus, by FTC-I

$$(F \circ g)(d) - (F \circ g)(c) = \int_c^d (F \circ g)'(x) \, dx
= \int_c^d f(g(x))g'(x) \, dx.$$  \hspace{1cm} (4)

Also, again by FTC-I,

$$
(F \circ g)(d) - (F \circ g)(c) = F(g(d)) - F(g(c))
= F(b) - F(a)
= \int_a^b f(x) \, dx.
$$  \hspace{1cm} (5)

Proof is complete from (4) and (5).

Theorems 17.2.1 and 17.2.2 give us techniques to evaluate definite integrals.

17.2.3 Examples:

(i) To evaluate

$$
\int x e^{ax} \, dx, \quad a \neq 0,
$$

we write

$$
F(x) = x, G(x) = e^{ax}.
$$

Then

$$
F'(x) = 1 \text{ and } G'(x) = ae^{ax}.
$$

Thus, by theorem 17.2.1,
\[ \int_{a}^{b} x e^{ax} \, dx = \frac{1}{\alpha} \int_{a}^{b} x (\alpha e^{ax}) \, dx \]
\[ = \frac{1}{\alpha} \left[ b e^{ab} - a e^{ab} \right] - \left[ \frac{e^{ax}}{\alpha} \right]_{a}^{b} \]
\[ = (b e^{ab} - a e^{ab}) - \frac{e^{ab} - e^{a}}{\alpha}. \]

(ii) To compute \[ I = \int_{0}^{1} 2x(x^2 + 1)^2 \, dx, \] let us write
\[ f(u) := \frac{1}{u^2}, u := g(x) = x^2 + 1. \]
Then by theorem 17.2.2,
\[ I = \int_{c}^{d} f(g(x)) g'(x) \, dx = \int_{a}^{b} \frac{1}{u^2} \, du, \]
where \( a = g(c) = 1 \) and \( b = g(d) = 2 \).
Hence
\[ I = \left[ \frac{2}{3} \frac{u^{-2}}{u} \right]_{1}^{2} = \frac{2}{3} \left[ \frac{2}{2^2 - 1} \right]. \]

17.2.4 Theorem (Leibnitz Rule):

Let \( f : [a, b] \to \mathbb{R} \) be continuous and \( u, v : [a, b] \to [a, b] \) be differentiable. Then \( \forall \gamma \in [a, b] \)
\[ \frac{d}{dx} \left( \int_{u(x)}^{v(x)} f(t) \, dt \right) \bigg|_{x=\gamma} = f(v(\gamma))v'(\gamma) - f(u(\gamma))u'(\gamma). \]

Proof:

Since for \( x \in [a, b] \),
\[ F(x) = \int_{a}^{x} f(t) \, dt \] is differentiable with \( F'(x) = f(x) \),
by chain rule, for all \( \gamma \in [a, b] \), we have
\[ \frac{d}{dx} (Fou) \bigg|_{x=\gamma} = F'\left(u(\gamma)\right)u'(\gamma) = f\left(u(\gamma)\right)u'(\gamma), \] (6)
and
\[ \frac{d}{dx} \left. (F \circ \nu) \right|_{x=\gamma} = F'(\nu(\gamma))\nu'(\gamma) = f(\nu(\gamma))\nu'(\gamma). \] (7)

Also by FTC-I,
\[ F(\nu(x)) - F(\mu(x)) = \int_{\mu(x)}^{\nu(x)} F'(t) \, dt = \int_{\mu(x)}^{\nu(x)} f(t) \, dt. \] (8)

Hence, by (6), (7) and (8), we have
\[ \left. \frac{d}{dx} \int_{\mu(x)}^{\nu(x)} f(t) \, dt \right|_{x=\gamma} = f(\nu(\gamma))\nu'(\gamma) - f(\mu(\gamma))\mu'(\gamma). \]

**17.2.5 Example:**

Let
\[ F(x) = \int_1^x \frac{1}{t} \, dt, \quad x > 0. \]

Then, by theorem 17.2.4, \( F'(x) \) exists and for \( x > 0 \),
\[ F'(x) = \frac{1}{x} - \left( \frac{1}{x^2} \right) = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}. \]

**PRACTICE EXERCISES**

1. Using Leibnitz’s Rule, compute the following:
   (a) \( \frac{d^2 y}{dx^2} \), if \( y = \int_0^x \frac{dt}{\sqrt{1+t^2}} \)
   (b) \( \frac{d}{dx} \), if for \( x \in \mathbb{R} \)
      (i) \( F(x) = \int_1^{2x} \cos(t^2) \, dt. \)
      (ii) \( F(x) = \int_0^{x^2} \cos(t) \, dt. \)

2. Let \( f: \mathbb{R} \to \mathbb{R} \) be continuous and \( \lambda \in \mathbb{R}, \lambda \neq 0 \). For \( x \in \mathbb{R} \), let
\[ g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t) \, dt. \]
Show that
\[ g(0) = 0 = g'(0) \]
and \( g \) satisfies the following:
\[ g''(x) + \lambda^2 g(x) = f(x) \text{ for all } x \in \mathbb{R}. \]
3. Let \( P \) be a real number and let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that

\[
f(x + p) = f(x) \quad \text{for all } x \in \mathbb{R}.
\]

Let

\[
\phi(x) := \int_{x}^{x+p} f(t) \, dt, \quad x \in \mathbb{R}.
\]

Show that \( \phi \) is a constant function, independent of \( P \).

4. Let \( f : [0, \infty) \to (0, \infty) \) a continuous function. For any \( b > 0 \), let \( G(b) \) denote the area bounded by the x-axis, the lines \( x = 0, x = b \) and the curve \( y = f(x) \). If, is given by

\[
G(b) = \sqrt{b^2 + 1} - 1 \quad \text{for each } b > 0,
\]

determine the function \( f \).

5. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Show that for every \( x \in [a, b] \),

\[
\int_{a}^{x} \left[ \int_{a}^{u} f(t) \, dt \right] \, du = \int_{a}^{x} (x-u) f(u) \, du.
\]

6. Integration by inverse substitution:

Let \( f : [a, b] \to \mathbb{R} \) and \( \phi : [c, d] \to [a, b] \) be such that the following are satisfied:

(i) \( f \) is continuous.

(ii) \( \phi \) is onto.

(iii) \( \phi' \) exists, is continuous on \( [c, d] \) and \( \phi'(y) \neq 0 \) for all \( y \in [c, d] \).

Show that \( \phi \) is one-one, and hence \( \phi^{-1} \) exists. Using direct substitution for \( \phi \), show that

\[
\int_{a}^{b} f(x) \, dx = \int_{\alpha}^{\beta} f(\phi(y)) \phi'(y) \, dy,
\]

where \( \alpha = \phi^{-1}(a) \) and \( \beta = \phi^{-1}(b) \).

7. Using direct/indirect substitution, compute the following:

(i) \[
\int \frac{2x}{\sqrt{9 + x^2}} \, dx,
\]

(use \( u = 9 + x^2 \)).

(ii) \[
\int \frac{dx}{\sqrt{9 + x^2}},
\]

(use inverse substitution).
Recap
In this section you have learnt the following

- Integration by parts formula
- Integration by substitution
- Leibnitz's formula for differentiating integral with variable limits

$x = 3 \tan \phi$