14.1 Taylor’s Theorem and its applications

In previous section we used \( L(x,a) \), the tangent line, a polynomial of degree one in \( x \) to approximate a given function \( f(x) \) for \( x \) near \( x = a \). One can try to approximate the function \( f \) by a higher degree polynomial, hoping that the polynomial of higher degree will give a better approximation to \( f \) for \( x \) near \( a \). To analyze this, we need a generalization of the extended mean value theorem:

14.1.1 Theorem (Taylor’s Theorem):

Let \( \delta > 0 \) and \( f : [a-\delta, a+\delta] \to \mathbb{R} \) be such that

(i) \( f, f', \ldots, f^{(n)} \) exist and are are continuous on \([a-\delta, a+\delta] \).

(ii) \( f^{(n+1)} \) exist on \([a-\delta, a+\delta] \).

Then,

\[
f(x) = f(a) + f'(a)(x-a)+\ldots+\left(\frac{f^{(n)}(a)}{n!}\right)(x-a)^n + \left(\frac{f^{(n+1)}(c)}{(n+1)!}\right)(x-a)^{n+1}.
\]

for some \( c \in (a-\delta, a+\delta) \)

The above expression is also known as the Taylor's formula for \( f \) around \( a \).
Proof:
We assume the proof. The interested reader may refer a book on advanced calculus.

14.1.2 Definition:

Let \( f \) be as in theorem 14.1.1.

(i) The polynomial

\[
T_n(x, a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.
\]

is called the \( n \)th degree Taylor polynomial of \( f \) around \( a \).

(ii) The term

\[
R_n(x, a) = f(x) - T_n(x, a) = \left( \frac{f^{(n+1)}(c)}{(n+1)!} \right)(x-a)^{n+1}
\]

is called the \( n+1 \)th Remainder term of the Taylor's formula for \( f \) around \( a \).

(iii) In case

\[
\lim_{n \to \infty} |R_n(x; a)| = 0,
\]

we say that \( f \) has Taylor series around \( a \) and write it as

\[
f(x) = \sum_{n=0}^{+\infty} \left( \frac{f^{(n)}(a)}{n!} \right)(x-a)^n.
\]

14.1.3 Example:

(i) Let

\[
f(x) = \frac{1}{x}, \quad x \neq 0.
\]

Then

\[
f^{(k)}(x) = \frac{(-1)^k}{x(k+1)}, \quad k \geq 1.
\]

Thus for any \( a > 0 \),
\( T_n(x, a) = f(a) - \frac{(x-a)}{a^2} + \frac{(x-a)^2}{a^3} - \ldots + \frac{(-1)^n (x-a)^n}{a^{n+1}}. \)

(ii) Let

\[ f(x) = \sin x, \quad x \in \mathbb{R} \]

Then, for \( k \in \mathbb{N} \),

\[ f^{(k)}(x) = \begin{cases} \frac{(-1)^{k/2}}{k!} \sin x, & \text{if } k \text{ is even} \\ \frac{(-1)^{(k-1)/2}}{k!} \cos x, & \text{if } k \text{ is odd} \end{cases} \]

Thus, for \( n \text{th} \) Taylor polynomial of \( f(x) \) around \( a = 0 \) is given by

\[ T_n(x; a) = \begin{cases} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^{(n-1)/2} x^n}{n!}, & \text{for } n \text{ odd} \\ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^{(n-2)/2} x^{n-1}}{(n-1)!}, & \text{for } n \text{ even} \end{cases} \]

(iii) Let \( f(x) \) be a polynomial of degree \( n \):

\[ f(x) = a_0 + a_1 x + \ldots + a_n x^n, \quad a_n \neq 0. \]

Then

\[ f^{(k)}(x) = \begin{cases} \frac{n(n-1)\ldots(n-k)}{k!} a_n x^{n-k}, & \text{for } 1 \leq k \leq n \\ 0, & \text{for } k > n \end{cases} \]

Thus, for \( f \),

\[ T_n(f; a) = T_{n+m}(f; a) \quad \text{for all } m \geq n. \]

For example, for

\[ f(x) = x^3 - 2x + 4, \]

\[ f'(x) = 3x^2 - 2, \quad f''(x) = 6x, \quad f'''(x) = 6 \quad \text{and} \quad f^{(iv)}(x) = 0 \]

Thus, for \( a = 2 \)

\[ T_3(x; 2) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 
= 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3 \]

14.1.4 Note:

For \( n = 1 \), the Taylor polynomial for \( f \) is nothing but the linear approximation. For a function \( f \), which is \( n + 1 \) times differentiable in \((a, b)\), we can use the \( n \text{th} \) degree Taylor's polynomial to approximate \( f \) in a small interval around \( a \). We analyze this for \( n = 2 \) in the next section.

Thus, for \( f \),
\[ T_n(f; a) = T_{n+m}(f; a) \text{ for all } m \geq n. \]

For example, for
\[ f(x) = x^3 - 2x + 4, \]
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Thus, for \( a = 2 \)
\[
T_3(x; 2) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3
= 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3
\]

14.1.4 Note:

For \( n = 1 \), the Taylor polynomial for \( f \) is nothing but the linear approximation. For a function \( f \), which is \( n + 1 \) times differentiable in \((a, b)\), we can use the \( n \)th degree Taylor's polynomial to approximate \( f \) in a small interval around \( a \). We analyze this for \( n = 2 \) in the next section.

**PRACTICE EXERCISES**

1. Find the \( n \)th Taylor polynomial of \( f \) around \( a \), that is,
   \[ f_n(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad x \in \mathbb{R}, \]
   when \( a = 0 \) and \( f(x) \) is as below:
   
   (i) \( x^3 + 3x^2 + 5 \).
   (ii) \( \tan^{-1}x \).
   (iii) \( \sin x \).
   (iv) \( \cos x \).
   (v) \( \exp(x) \).
   (vi) \( \exp(-x) \).

2. Show that for all \( x \in \mathbb{R} \),
   
   (i) \( \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \).
   (ii) \( \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \).
   (iii) \( \exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

3. Let \( T_n(f; a) \) denote the Taylor's polynomial of order \( n \) for \( f \) at \( x = a \). Prove the following:
   
   (i) \( T_n(af; a) = aT_n(f; a) \) for every \( a \in \mathbb{R} \).
   (ii) \( T_n(f+g; a) = T_n(f; a) + T_n(g; a) \).
   (iii) \( \ldots \).
\[ (T_n(f; a))' = T_{n-1}(f'; a) \]

**PRACTICE EXERCISES**

4. Let \( f, g \) be functions having derivatives of order \( n \) at \( x = a \) such that

\[ f(x) = P_n(x) + x^n g(x), \]

where \( P_n(x) \) is a polynomial of degree \( n \geq 1 \). If \( g(x) \to 0 \) as \( x \to 0 \), show that

\[ P_n(x) = T_n(x; 0). \]

5. Using exercises (3) and (4) above find \( T_n(f, 0) \) for the following:

\( f(x) = \frac{1}{1-x} \)

\[ \text{(ii)} \quad f(x) = \frac{1}{1+x}. \]

\( f(x) = \frac{1}{1-x^2} \)

\[ \text{(iii)} \quad f(x) = \frac{1}{1-x^2} \left( \text{Hint: } \frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x} \right) \]

\( f(x) = \frac{1}{1+x^2} \)

\[ \text{(iv)} \quad f(x) = \frac{1}{1+x^2}. \]

\( f(x) = \frac{x}{1-x^2} \)

\[ \text{(v)} \quad f(x) = \frac{x}{1-x^2}. \]

6. Using exercise (3) and 5(iv), show that for \( f(x) = \tan^{-1}(x) \),

\[ T_4(f, 0) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}. \]

**Recap**

In this section you have learnt the following

- Taylor's theorem and its applications

**[Section 14.2]**

**Objectives**

In this section you will learn the following:

- How to estimate a function by a quadratic function and how to estimate the error.
14.2 Quadratic approximations

14.2.1 Definitions:

For \( n = 2 \), the Taylor's polynomial

\[
T_n(x; a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2,
\]

is called the quadratic approximation of \( f \) for \( x \) near \( a \), and is also denoted by \( Q(x; a) \).

14.2.2 Examples:

(i) Consider the function

\[
f(x) = \frac{1}{1-x} \quad \text{for } x \neq 1.
\]

Then,

\[
f(x) = \frac{1}{(1-x^2)} \quad \text{and} \quad f''(x) = \frac{-2}{(1-x)^3} \quad \text{for } x \neq 1
\]

Thus, for \( x \) near \( a \neq 1 \),

\[
Q(x; a) = \frac{1}{1-a} + \frac{1}{(1-a)^2}(x-a) + \frac{-2(x-a)^2}{2(1-a)^3}.
\]

For example for \( a = 0 \),

\[
Q(x; 0) = 1 + x + x^2.
\]

(ii) Consider the function

\[
f(x) = \exp(x), \ x \in \mathbb{R}.
\]

Then

\[
f(0) = f'(0) = f''(0) = \exp(0) = 1.
\]

Thus, near \( a = 0 \), \( f(x) \) has quadratic approximation...
\[ Q(x; 0) = 1 + x + \frac{x^2}{2} \]

(iii) Let

\[ f(x) = \sqrt{x}, \ x \in \mathbb{R}, \ x > 0 \]

Then,

\[ f'(x) = \frac{1}{2\sqrt{x}} \quad \text{and} \quad f''(x) = -\frac{1}{4(x)^{3/2}} \]

Thus, near \( x = 1 \), \( f(x) \) has quadratic approximation:

\[ Q(x, 1) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2} (x - 1)^2 \]

\[ = 1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{8}. \]

For example, this gives

\[ \sqrt{0.5} = 1 - \left( \frac{0.5}{2} \right) - \frac{(0.5)^2}{8} \]

\[ = 0.719. \]

Like for linear approximations, it is natural to ask the question:

How well does \( Q(x, a) \) approximate \( f(x) \) for near \( a \)?

An answer to this question is the following:

14.2.3 Corollary:

Let \( \alpha, \beta \in \mathbb{R} \) with \( \alpha \neq x \) and \( J \) be the closed interval with end points \( \alpha \) and \( x \). Let \( f : I \to \mathbb{R} \) be such that

(i) The functions \( f, f', f'' \) are all continuous.

(ii) For every \( c \) between \( \alpha \) and \( x \),

\[ f''(c) \text{ exists and } |f''(c)| \leq M_3(x), \]

then \( e_2(x, a) = f(x) - Q(x, a) \) satisfies the following:

\[ |e_2(x, a)| \leq \left( \frac{M_3(x)}{3!} \right) |x-a|^3. \]

**Proof**

14.2.3 Corollary:

Let \( \alpha, \beta \in \mathbb{R} \) with \( \alpha \neq x \) and \( J \) be the closed interval joining \( \alpha \) and \( x \). Let \( f : I \to \mathbb{R} \) be such that

(i) The functions \( f, f', f'' \) are all continuous.

(ii) For every \( c \) between \( \alpha \) and \( x \),

\[ f''(c) \text{ exists and } |f''(c)| \leq M_3(x), \]

then \( e_2(x, a) = f(x) - Q(x, a) \) satisfies the following:

\[ |e_2(x, a)| \leq \left( \frac{M_3(x)}{3!} \right) |x-a|^3. \]
Proof:
Follows trivially from Taylor's Theorem for \( n = 3 \).

14.2.4 Example:

Consider the function

\[ f(x) = \frac{1}{1-x} \text{ for } x \neq 1. \]

Then

\[ f^{(n)}(c) = \frac{6}{(1-c)^4} \text{ for } c \neq 1. \]

We saw in example 14.2.1, that the quadratic approximation for \( f \) near the point \( a = 0 \) is given by

\[ Q(x; 0) = 1 + x + x^2. \]

Let us estimate the error

\[ e_2(x; 0) = f(x) - Q(x; 0) \text{ for } x < 1. \]

Let us fix \( x < 1 \). We have to consider two cases.

**Case (i):** \( 0 < x < 1 \).

In this case, for \( 0 < c < x \), we have

\[ |f^{(n)}(c)| = \frac{6}{(1-c)^4} \leq \frac{1}{(1-x)^4} = M_3(x). \]

Thus

\[ |e_2(x, 0)| \leq \left( \frac{|x|^3}{(1-x)^4} \right). \]

For example,

for all \( 0 < x < 0.1, |e_2(x, 0)| \leq |x|^3 \leq (0.1)^3 = 0.001. \]

Thus, for all \( x \in [0, 0.1] \),

\[ Q(x, 0) = 1 + x + x^2 \text{ differs from } f(x) = \frac{1}{1-x} \text{ at most by } 0.001. \]

**Case (ii):** \( x < 0 \).

In this case, for \( x < c < 0 \), we have

\[ |f^{(n)}(c)| = \frac{6}{(1-c)^4} < 6 = M_3(x). \]

Thus,

\[ |e_2(x, 0)| \leq |x|^3. \]

For example,

for \( x = -0.01 < 0, |e_2(x, 0)| \leq (0.01)^3 = 0.001. \]

14.2.5 Note (Rate of convergence for the error):

Since

\[ f(x) = Q(x; a) + e_2(x; a), \]
clearly, \( e_2(x; a) \to 0 \) as \( x \to a \). In fact, the error in the quadratic approximation tends to zero at a rate faster than \( (x-a)^2 \). To see this, note that

\[
e_2(x; a) := \frac{(x-a)^3}{3!} f'''(c),
\]

and hence

\[
\lim_{x \to a} \left( \frac{e_2(x; a)}{(x-a)^2} \right) = \lim_{x \to a} \left( \frac{f'''(c)(x-a)}{3!} \right) = 0.
\]

**PRACTICE EXERCISES**

1. Let \( a \in \mathbb{R} \) and \( F(x) = c_0 + c_1(x-a) + c_2(x-a)^2 \) for \( x \in \mathbb{R} \). If \( f \) is twice differentiable at \( a \), show that \( f \) is the quadratic approximation of \( f \) near \( a \) if and only if

\[
\begin{align*}
    f(a) &= F(a) = c_0, \\
    f'(a) &= F'(a) = c_1, \quad \text{and} \\
    f''(a) &= F''(a) = c_2.
\end{align*}
\]

2. Let

\[
f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} \quad \text{for} \quad x > 0.
\]

Write down the linear and the quadratic approximations \( L(x) \) and \( Q(x) \) of \( f(x) \) near the point \( x = 4 \). Estimate the errors.

3. For the following functions find the quadratic approximations near the point \( a = 0 \), and also find the error bounds valid for \( |x| \leq 0.03 \):

   (i) \( f(x) = \sin(x^2) \).
   (ii) \( f(x) = \sqrt[3]{1+x} \).
   (iii) \( f(x) = \frac{1}{\sqrt[3]{1+x}} \).
   (iv) \( f(x) = \frac{x}{1+x^2} \).

**Recap**

In this section you have learnt the following
How to estimate a function by a quadratic function and how to estimate the error.