Semiconductor Optical Communication Components and Devices

Lecture 13: Slab Waveguides

Prof. Utpal Das
Professor, Department of Electrical Engineering,
Laser Technology Program,
Indian Institute of Technology, Kanpur

http://www.iitk.ac.in/ee/faculty/det_resume/utpal.html
Let us assume that \( n_f > n_s \geq n_c \)

\[
E(r,t) = E(x,y) e^{i(\omega t - \beta z)}
\]

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E(x,y) + \left[ k_o^2 n_c^2 (r) - \beta^2 \right] E(x,y) = 0
\]

Putting \( \partial/\partial x = 0 \) and writing it separately in region I, II and III yields

**Region I**

\[
\frac{\partial^2}{\partial y^2} E(x,y) + \left( k_o^2 n_c^2 (r) - \beta^2 \right) E(x,y) = 0
\]

**Region II**

\[
\frac{\partial^2}{\partial y^2} E(x,y) + \left( k_o^2 n_f^2 (r) - \beta^2 \right) E(x,y) = 0
\]

**Region III**

\[
\frac{\partial^2}{\partial y^2} E(x,y) + \left( k_o^2 n_s^2 (r) - \beta^2 \right) E(x,y) = 0
\]

\[ V = k_o d \sqrt{n_f^2 - n_c^2} = \frac{2\pi}{\lambda_o} d \sqrt{n_f^2 - n_c^2} \]

\[ b = \frac{\beta^2 - n_c^2 k_o^2}{k_o^2 (n_f^2 - n_c^2)} = \frac{N^2 - n_c^2}{n_f^2 - n_c^2} \]

\[ \alpha_c^2 = \beta^2 - k_o^2 n_c^2 \]

\[ \alpha_s^2 = \beta^2 - k_o^2 n_s^2 \]

\[ k_{f_y}^2 = k_o^2 n_f^2 - \beta^2 \]
Three Layer Slab Waveguide

For TE modes: Field equation for the three layers

\[
E_x = \begin{cases} 
A \exp(-\gamma_c y) & y > 0 \\
B \exp(jk_y y) + C \exp(-jk_y y) & 0 > y > -d \\
D \exp(\gamma_s y) & y < -d
\end{cases}
\]

where \( \gamma_c^2 = \beta^2 - n_c^2 k_o^2 \), \( k_y^2 = n_f^2 k_o^2 - \beta^2 \), \( \gamma_s^2 = \beta^2 - n_s^2 k_o^2 \)

Applying boundary condition (i) at \( y=0 \)

\[ A = B + C \quad \text{at} \quad y=0 \quad (1) \]

For continuity of \( \partial E_x / \partial y \) at \( y=0 \)

\[-\gamma_c A = jk_y B - jk_y C \quad (2)\]

(ii) at \( y=-d \) continuity of \( E_x \) at \( y=-d \) we get

\[ B \exp(-jk_y d) + C \exp(jk_y d) = D \exp(-\gamma_s d) \quad (3) \]

continuity of \( \partial E_x / \partial y \) at \( y=-d \)

\[ jk_y B \exp(-jk_y d) - jk_y C \exp(jk_y d) = \gamma_s D \exp(-\gamma_s d) \quad (4) \]

Solving (1) and (2), we get

\[-\gamma_c (B+C) = jk_y (B-C) \]
\[-\gamma_c B - \gamma_c C - jk_y B + jk_y C = 0 \]
\[(\gamma_c + jk_y)B + (\gamma_c - jk_y)C = 0 \]
Solving (3) and (4), we get
\[ \gamma_s B \exp(-jk_y d) + \gamma_s C \exp(jk_y d) = \gamma_s D \exp(-\gamma_s d) \]
\[ jk_y B \exp(-jk_y d) - jk_y C \exp(jk_y d) = \gamma_s D \exp(-\gamma_s d) \]
\[ (\gamma_s - jk_y) B \exp(-jk_y d) + (\gamma_s + jk_y) C \exp(jk_y d) = 0 \quad (6) \]

Solving (5) and (6), we get
\[ B = - \left[ (\gamma_c - jk_y) C / [\gamma_c + jk_y] \right] \quad \text{from equation (5)} \]
and \[ B = - \left[ (\gamma_s + jk_y) \exp(jk_y d) C / [(\gamma_s - jk_y) \exp(-jk_y d)] \right] \quad \text{from equation (6)} \]
\[ \Rightarrow - [(\gamma_s + jk_y) \exp(jk_y d)] / [(\gamma_s - jk_y) \exp(-jk_y d)] = (\gamma_c - jk_y) / (\gamma_c + jk_y) \]
\[ \Rightarrow \exp(jk_y d) / \exp(-jk_y d) = \left\{ [(\gamma_c - jk_y) (\gamma_s - jk_y)] / [(\gamma_c + jk_y) (\gamma_s + jk_y)] \right\}. \]
\[ 2jk_y d = \log[(\gamma_c - jk_y)^2(\gamma_s - jk_y)^2] - \log[(\gamma_c^2 + k_y^2)(\gamma_s^2 + k_y^2)] \]

Equation the imaginary parts, we get by using \( \log(x+jy) = \sqrt{x^2 + y^2} + j\tan^{-1}(y/x) \)
\[ k_y d = m\pi + \tan^{-1}(-k_y/\gamma_c) + \tan^{-1}(-k_y/\gamma_s) \]
For \( h=2d \), \( k_y 2d = m\pi + \tan^{-1}(-k_y/\alpha) + \tan^{-1}(-k_y/\alpha_s) \), \( V = k_o 2d (n_f^2 - n_s^2)^{1/2} \)
\[ k_y d = m\pi + \tan^{-1}(\gamma_c/k_y) + \tan^{-1}(\gamma_c/k_y) \]
\[ a = (n_s^2 - n_c^2) / (n_f^2 - n_s^2), \quad V = k_o d (n_f^2 - n_s^2)^{1/2}, \quad b = (n_{eff}^2 - n_s^2) / (n_f^2 - n_s^2) \]
\[ n_{eff} = \beta / k_o \text{ and } k_o = 2\pi/\lambda \]

Then above equation can be written as
\[ V(1-b)^{1/2} = m\pi + \tan^{-1}[b/(1-b)]^{1/2} + \tan^{-1}[(b+a)/(1-b)]^{1/2} \]
For the equation of the film
\[ E_x = B \exp(jk_y y) + C \exp(-jk_y y) \] can be written in the form of either
\[ E_x = A \cos(k_y y) \text{ or } k_x = B \sin(k_y y) \]

I. Now, taking \( E_x = A \cos(k_y y) \)
\[ E_x = \begin{cases} 
C \exp(-\alpha_f y) & y > 0 \\
A \cos(k_y y) & 0 > y > -d \\
B \exp(\alpha_s y) & y < -d 
\end{cases} \]

Continuity of \( E_x \) at \( y=0 \) gives \( C = A \) and continuity \( \partial E_x / \partial y \) at \( y=0 \) gives \( -\alpha_f C = k_y A \) or \( -\alpha_f = k_y \) \( \text{(ii)} \)

Continuity of \( E_x \) at \( y=-d \) gives \( A \cos(k_y d) = B \exp(-\alpha_s d) \) \( \text{(iii)} \)

Continuity of \( \partial E_x / \partial y \) at \( y=-d \) gives \( k_y A \sin(k_y d) = \alpha_s B \exp(-\alpha_s d) \) \( \text{(iv)} \)

Dividing (iv) by (iii) we have \( k_y \tan(k_y d) = \alpha_s \), \( \Rightarrow \tan(k_y d) = \alpha_s / k_y \)

\[ k_y d = \tan^{-1}(\alpha_s / k_y) \] \( \text{(v)} \)

II. Now, for \( E_x = C \exp[-\alpha_f (y-d/2)], = B \sin[k_y (y-d/2)], \text{ and } = B \exp[\alpha_s (y+d/2)] \) for the respective regions.

Solving similarly, we get \( k_y d = \tan^{-1}(-k_y / \alpha_s) \) \( \text{(vi)} \)

From (v) and (vi) we get that
\[ \tan^{-1}(\alpha_s / k_y) = \tan^{-1}(-k_y / \alpha_s) \]
Symmetric Slab Waveguide \( n_f > n_s \rightarrow (t_g = 2d) \)

**TE** : Even Mode: \( \tan(k_{fy}d) = \frac{\alpha_s}{k_{fy}} \), Odd Mode: \( \tan(k_{fy}d) = -\frac{k_{fy}}{\alpha_s} \)

\[
\tan(2k_{fy}d) = \tan(k_{fy}t_g) = \frac{2\tan(k_{fy}d)}{1-\tan^2(k_{fy}d)}
\]

Even Mode:

\[
\tan(k_{fy}t_g) = \frac{2\alpha_s/k_{fy}}{1-(\alpha_s/k_{fy})^2} = \frac{2\alpha_s k_{fy}}{k_{fy}^2 - \alpha_s^2}
\]

Odd Mode:

\[
\tan(k_{fy}t_g) = \frac{2(-k_{fy}/\alpha_s)}{1-(k_{fy}/\alpha_s)^2} = \frac{2\alpha_s k_{fy}}{k_{fy}^2 - \alpha_s^2}
\]

**TM** : can be shown to give

\[
\tan(k_{fy}t_g) = \frac{2\overline{\alpha}_s k_{fy}}{k_{fy}^2 - \overline{\alpha}_s^2}
\]

where \( \overline{\alpha}_s = (n_f^2/n_s^2)\alpha_s \)

\[
V = \left[ k_f^2 - k_s^2 \right]^{1/2}d = \left[ \alpha_s^2 + k_{fy}^2 \right]^{1/2}d
\]

where \( k_{fy}^2 = k_f^2 - \beta^2; \ \alpha_s^2 = \beta^2 - k_s^2 \)

**FOR GUIDED MODES** \( k_s < \beta < k_f \)
Symmetric Slab Waveguide \( n_f > n_s \rightarrow (t_g = 2d) \)

For lower modes \( \beta \) closer to \( k_f \), for higher modes \( \beta \) closer to \( k_s \).

For \( \beta \) closer to \( k_f \), \( k_{fy} \) is small and \( \alpha_s \) is large, for \( \beta \) closer to \( k_s \), \( k_{fy} \) is large and \( \alpha_s \) is small.

For \( \beta < k_s \), \( \alpha_s \) is imaginary, thus in the cladding we do not have evanescent tail.

\[
k^2 = k_x^2 + k_y^2 + k_z^2 = 0 + k_y^2 + \beta^2
\]

Therefore, for \( k_y \) large (higher order modes) \( \beta \) is smaller.
Symmetric Slab Waveguide \( n_f > n_s \rightarrow (t_g = 2d) \)

At cut-off (when \( \alpha_s \) goes from real to imaginary), \( \alpha_s = 0 \)

\[
\beta = k_s \quad \text{and} \quad k_{fy}^2 = k_f^2 - k_s^2 = (2\pi/\lambda)^2 (n_f^2 - n_s^2) \quad \text{and} \quad \tan(k_{fy} t_g) = 0
\]

\[
\therefore k_{fy} t_g = m_s \pi \quad \text{for} \quad m_s = 0, 1, 2, \ldots
\]

or \( (2\pi/\lambda)^2 (n_f^2 - n_s^2) t_g^2 = m_s^2 \pi^2 \)

\[
\therefore n_f - n_s = \frac{\lambda^2 m_s^2}{4(n_f + n_s) t_g^2} \quad \text{or} \quad \Delta n \geq \frac{\lambda^2 m_s^2}{4(n_f + n_s) t_g^2}
\]

For \( m_s \)th mode to propagate

Therefore, \( \Delta n \) larger means \( m_s \) is larger, \( k_{fy} \) is larger, \( \alpha_s \) is lower

**Note:** The mode \( m_s = 0 \) does not have a cut-off i.e., there is no non-zero value of \( \Delta n \) or \( t_g \) for which \( m_s = 0 \)th mode will not propagate. No cut-off with \( \lambda \) also, \( \lambda = \infty \) for \( \Delta n \) to be infinite.
Slab Waveguides

Asymmetric slab

For Waveguiding:
\[ n_{\text{film}} > n_{\text{substrate}} \gg n_{\text{cladding}} \]

Symmetric:
\[ n_{\text{substrate}} = n_{\text{cladding}} \]
Asymmetric Slab Waveguide \( n_f > n_s >> n_c \)

\[
k_{fy}^2 = k_f^2 - \beta^2
\]

\[
\alpha_s^2 = \beta^2 - k_s^2
\]

\[
\alpha_c^2 = \beta^2 - k_c^2
\]

It can be shown that

For TE \( \tan(k_{fy}t_g) = \frac{(\alpha_s + \alpha_c)k_{fy}}{k_{fy}^2 - \alpha_s \alpha_c} \)

For TM \( \tan(k_{fy}t_g) = \frac{(\bar{\alpha}_s + \bar{\alpha}_c)k_{fy}}{k_{fy}^2 - \bar{\alpha}_s \bar{\alpha}_c} \)

where \( \bar{\alpha}_s = (n_f^2/n_s^2)\alpha_s; \bar{\alpha}_c = (n_f^2/n_c^2)\alpha_c \)

Therefore, Modes leak power into the II layer (film) when \( \alpha_s \) goes from real to imaginary. 

For \( \alpha_s \) going from real to imaginary there is absolutely no guiding

Since \( n_s >> n_c \) for guided modes (i.e., \( k_s < \beta < k_f \)) \( \alpha_c >> \alpha_s \)

No guiding when \( \alpha_c \) is here

Leaky waveguide: Substrate Radiation when \( \alpha_s \) is here

Actually for good waveguides \( \alpha_s = 0 \) is still the condition considered for no guiding (or cut-off)
Consider modes of the symmetric waveguide for which the intensity goes to zero at $y = 0$ (i.e., $m_S = 1,3,5,7,\ldots$). The slope of the intensities at $y = 0$ are close to those of the asymmetric waveguide at $y = d$ when $n_f >> n_s$.

Approximation done by the Dotted Curve

Modes of the asymmetric waveguides are similar to the odd modes in half the waveguide of the symmetric waveguide.
Asymmetric Slab Waveguide \( n_f > n_s \gg n_c \)

Therefore, Modes of the asymmetric waveguide ‘\( m_a \)’, \( m_a = 0,1,2,\ldots \) should be related to the odd modes of the symmetric \( m_S = (2m_a + 1) \), \( m_S = 1,3,5,\ldots \).

Therefore, using the condition \( \alpha_s = 0 \) for cut-off

\[
n_f - n_s \geq \frac{m_S^2 \lambda^2}{4(n_f + n_s)t^2_{gsymmetric}}
\]

But since the asymmetric waveguide modes resemble the symmetric mode in half the waveguide

\[
t_{gsymmetric} = 2t_{gasymmetric}
\]

or \( n_f - n_s \geq \frac{(2m_a + 1)^2 \lambda^2}{4(n_f + n_s)(2t_{gasymmetric})^2}
\]

Note: even \( m_a = 0 \) has a cut-off
Normalized Propagation constant as a function of Normalized Frequency

Fundamental mode in slab waveguides for different asymmetry factors

Normalized Propagation Constant 'b'

Normalized Frequency 'V'
The normalized waveguide index parameter $b$ is given by:

$$V \sqrt{(1-b)} = m \pi + \tan^{-1} \left( \frac{b}{(1-b)} \right) + \tan^{-1} \left( \frac{(b+a)}{(1-b)} \right)$$

The normalized frequency / Slab thickness parameter $V$ is:

$$V = kd \sqrt{n_f^2 - n_s^2}$$

The parameter $b$ is:

$$b = \frac{n_{\text{eff}}^2 - n_{\text{sub}}^2}{n_{\text{film}}^2 - n_{\text{sub}}^2}$$

The waveguide parameters are:

$$k_o = \frac{2\pi}{\lambda} \quad n_{\text{eff}} = \frac{\beta}{k_o}$$

The effective indices are:

$$n_f = n_{\text{film}}$$

$$n_c = n_{\text{cladding}}$$

$$n_s = n_{\text{substrate}}$$

The parameter $a$ is:

$$a = \frac{n_s^2 - n_c^2}{n_f^2 - n_s^2}$$
1. For a planar GaAs waveguide of thickness 0.5 μm and refractive indices $n_f = 3.5$, $n_s = 3.45$, and $n_c = n_o = 1.0$, find a value of the thickness $t_g$ for which there would be propagation of only the lowest order mode for a wavelength of 1.5 μm but with the maximum propagation constant possible. Calculate this propagation constant. Find the cutoff wavelength for the lowest order mode in this waveguide.

2. A five layer slab waveguide structure composed of layers $n_o = 1.000$, $t_{go} =$ semi-infinite; $n_1 = 3.498$, $t_{g1} = 1.5$ mm; $n_f = 3.500$, $t_{gf} = 2.5$ mm; $n_2 = 3.495$, $t_{g2} = 1.5$ mm; and $n_s = 3.5$, $t_{gs} =$ semi-infinite is supposed to guide TE light at a wavelength $\lambda$. Find the range of wavelengths for which there would be single mode propagation. [Hint: Write the wave equation, i.e. the field expressions in the five layers for the field components $E_z$ and $H_z$. Match the boundary conditions at the four interfaces. Then use Matlab or any other suitable software to solve the equations and find the relationship of $b$ vs $V$. Define $b$ and $V$ with respect to $n_f$ and $t_{gf}$. Get your answer from the plot.]