Butterworth Filter

If \( H(s) \) is Transfer Function \( \frac{V_o(s)}{V_{in}(s)} \)
given by \( H(s) = \frac{A(s)}{B(s)} \)
And if this has only poles but no zeros \( A(s) \) cost = \( H_o \)
Then \( H(s) = \frac{H_o}{B(s)} \)
Then the \( F^{th} \) 
\( B^2(\omega) = 1 + \epsilon^2 (\frac{\omega}{\omega_o})^{2F} \) is called
Butterworth Polynomial
The filters using Butterworth \( F^{th} \) are called
'Maximally Flat' kind, i.e. Ripple is v. low.
\[ |H(j\omega)| = \frac{H_0}{\sqrt{1 + \varepsilon^2 \left(\frac{\omega}{\omega_0}\right)^2}} \]

At \( \omega = \omega_0 \), \( H(j\omega) = \frac{H_0}{\sqrt{1 + \varepsilon^2}} \)

\varepsilon \) gives measure of Maximum Transmission \( A_{\text{max}} \)

\[ A_{\text{max}} = 20 \log \left(1 + \varepsilon^2\right)^{1/2} \]

\( \gamma \) \( \varepsilon = \sqrt{10^{A_{\text{max}}/10} - 1} \) gives Max. Transmission

If \( \varepsilon = 1 \)

\[ \left| \frac{H(j\omega)}{H_0} \right|^2 = \frac{1}{2} \quad \Delta \text{A}_{\text{max}} = 0 \quad \text{Passband terminates at} \ \omega = \omega_0 \]
At \( \omega = \omega_0 \),

\[
\left| \frac{H(j\omega)}{H_0} \right| = -20 \log \left[ \frac{1}{\sqrt{1 + e^{2(\omega/\omega_0)^2N}}} \right]
\]

\[
= 10 \log \left[ 1 + e^{2 \left( \frac{\omega}{\omega_0} \right)^2N} \right]
\]

\therefore \text{ Larger } N \text{ means larger attenuation.}
Example: For a LP Butterworth filter, we need attenuation of 40 dB and at $\frac{\omega}{\omega_0} = 2$. We use $\epsilon = 1$.

Then

$$|\frac{H(j\omega)}{H_0}|^2 = \frac{1}{1 + (\frac{\omega}{\omega_0})^{2N}}$$

Given $\frac{H(j\omega)}{H_0} = \frac{1}{100} = 0.01$

$$10^{-4} = \frac{1}{1 + 2^{2N}} \Rightarrow 2^{2N} = 10^4 - 1 \approx 10^4$$

$$2N \log_2 2 = 4 \Rightarrow N = \frac{2}{\log_2 10} = \frac{2}{0.3010} \approx 6.64 \approx 7$$
Chbyshev Polynomial

\[ C_n(\omega) = \cos(N \cos^{-1}(\frac{\omega}{\omega_0})) = \cosh(N \cosh^{-1}(\frac{\omega}{\omega_0})) \quad \omega \geq \omega_0 \]

Given \[ \left| \frac{H(j\omega)}{H_0} \right|^2 = (\text{40 dB})^{-1} = 10^{-4} \]

\[ 10^{-4} = \frac{1}{1 + (0.5084)^2 C_N^2(2)} \]

\[ C_N^2(2) = \frac{10^4 - 1}{(0.5084)^2} = 3.86 \times 10^4 \]

\[ C_N(2) = \sqrt{3.86 \times 10^2} \]

\[ \omega = 196.5 = \cosh(N \cosh^{-1}(2)) \]

Solving, \[ N = 4.53 \approx 5 \]
The ripple frequency $\omega_c$ is related to `-3db' cut-off frequency $\omega_0$ as

$$\omega_c = \omega_0 \cosh \left( \frac{1}{N} \cosh^{-1} \frac{1}{\xi} \right)$$

For 1db ripple with say $N=5$, $\omega_H = 1.03 \omega_c$

Example: $\gamma = 1$db, so $\frac{\omega}{\omega_0} = 2$; Attenuation is 40 db.

Since $\gamma = 1$db, $\xi = 0.5089$

$$\therefore \left| \frac{H(j\omega)}{H_0} \right|^2 = \frac{1}{1 + (0.5089)^2 C_n^2 \gamma^2}$$
Chebyshev filters are 'All Pole' filters and has larger 'Ripple' but sharper fall for lower Number of Sections \( N \) compared to Butterworth. 

\( N \) represents number of poles.

Parameter \( \varepsilon \) is related to Passband Ripple \( \gamma \) in \( \text{d} \text{B} \) by

\[
\varepsilon^2 = 10^{\gamma/10} - 1
\]

For 0.5 dB Ripple \( (\gamma = 0.5) \), \( \varepsilon = 0.3493 \)

\( \varepsilon \) for 1.0 dB Ripple \( (\gamma = 1.0) \), \( \varepsilon = 0.5089 \)
The Chebyshev Filters

If the Transfer $F^n$ has the form as:

$$|H(\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 \cos^2 \left[ N \cos^{-1} \left( \frac{\omega}{\omega_0} \right) \right]}} \quad \text{for} \quad \omega \leq \omega_0$$

$$= \frac{1}{\sqrt{1 + \epsilon^2 \cosh^2 \left[ N \cosh^{-1} \left( \frac{\omega}{\omega_0} \right) \right]}} \quad \text{for} \quad \omega \geq \omega_0$$

Then Transfer $F^n$ represents Chebyshev Function
\[ \frac{\omega_H}{\omega_C} = \cosh\left(\frac{1}{2}\right) \cosh^{-1} \frac{1}{0.5089} \]

\[ = 1.03 \]

Clearly we have a sharper fall at \( \omega_H \) to \( \omega_C \) with \( N = 5 \) and \( \gamma = 1.0 \) dB.
### Butterworth Polynomial

<table>
<thead>
<tr>
<th>n</th>
<th>Factors of polynomial B_n(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((s + 1))</td>
</tr>
<tr>
<td>2</td>
<td>((s^2 + 1.414s + 1))</td>
</tr>
<tr>
<td>3</td>
<td>((s + 1)(s^2 + s + 1))</td>
</tr>
<tr>
<td>4</td>
<td>((s^2 + 0.765s + 1)(s^2 + 1.848s + 1))</td>
</tr>
<tr>
<td>5</td>
<td>((s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1))</td>
</tr>
<tr>
<td>6</td>
<td>((s^2 + 0.518s + 1)(s^2 + 1.414s + 1)(s^2 + 1.932s + 1))</td>
</tr>
<tr>
<td>7</td>
<td>((s + 1)(s^2 + 0.444s + 1)(s^2 + 1.247s + 1)(s^2 + 1.802s + 1))</td>
</tr>
<tr>
<td>8</td>
<td>((s^2 + 0.399s + 1)(s^2 + 1.11s + 1)(s^2 + 1.663s + 1)(s^2 + 1.962s + 1))</td>
</tr>
</tbody>
</table>

### Chebyshev Polynomial

- **0.5-dB ripple \( \varepsilon = 0.3493 \)**

<table>
<thead>
<tr>
<th>n</th>
<th>Factors of polynomial C_n(s)</th>
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<tbody>
<tr>
<td>1</td>
<td>((s + 1.863))</td>
</tr>
<tr>
<td>2</td>
<td>((s^2 + 1.425s + 1.516))</td>
</tr>
<tr>
<td>3</td>
<td>((s + 0.626)(s^2 + 0.626s + 1.142))</td>
</tr>
<tr>
<td>4</td>
<td>((s^2 + 0.351s + 1.064)(s^2 + 0.845s + 0.356))</td>
</tr>
<tr>
<td>5</td>
<td>((s + 0.362)(s^2 + 0.224s + 1.036)(s^2 + 0.586s + 0.477))</td>
</tr>
<tr>
<td>6</td>
<td>((s^2 + 0.1556s + 1.024)(s^2 + 0.414s + 0.5475)(s^2 + 0.5796s + 0.157))</td>
</tr>
<tr>
<td>7</td>
<td>((s + 0.2522)(s^2 + 0.1014s + 1.015)(s^2 + 0.3194s + 0.6657)(s^2 + 0.4616s + 0.2539))</td>
</tr>
<tr>
<td>8</td>
<td>((s^2 + 0.0872s + 1.012)(s^2 + 0.2494s + 0.7413)(s^2 + 0.3718s + 0.3872)(s^2 + 0.4386s + 0.08005))</td>
</tr>
</tbody>
</table>

- **1.0-dB ripple \( \varepsilon = 0.5089 \)**

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<th>Factors of polynomial C_n(s)</th>
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<tbody>
<tr>
<td>1</td>
<td>((s + 1.965))</td>
</tr>
<tr>
<td>2</td>
<td>((s^2 + 1.098s + 1.103))</td>
</tr>
<tr>
<td>3</td>
<td>((s + 0.494)(s^2 + 0.494s + 0.994))</td>
</tr>
<tr>
<td>4</td>
<td>((s^2 + 0.279s + 0.987)(s^2 + 0.674s + 0.279))</td>
</tr>
<tr>
<td>5</td>
<td>((s + 0.289)(s^2 + 0.179s + 0.988s + 0.468s + 0.429))</td>
</tr>
<tr>
<td>6</td>
<td>((s^2 + 0.1244s + 0.9907)(s^2 + 0.3998s + 0.3577)(s^2 + 0.4642s + 0.1247))</td>
</tr>
<tr>
<td>7</td>
<td>((s + 0.2054)(s^2 + 0.0914s + 0.9927)(s^2 + 0.2562s + 0.6535)(s^2 + 0.3702s + 0.2304))</td>
</tr>
<tr>
<td>8</td>
<td>((s^2 + 0.07s + 0.9942)(s^2 + 0.1994s + 0.7236)(s^2 + 0.2994s + 0.3408)(s^2 + 0.3518s + 0.0702))</td>
</tr>
</tbody>
</table>
Sallan-Key Low Pass Section

Non Inverting

Inverting
Creation of a Real Single Pole

\[ \frac{V_o}{V_{in}} = H(s) = \frac{1}{1+RCS} = \frac{1}{1+s/(1/RC)} \rightarrow \text{NonInv.} \]

\[ \frac{V_o}{V_{in}} = -\left(\frac{R_2}{R_1}\right) \cdot \frac{1}{1+s/(1/RC)} \rightarrow \text{Inverting} \]
Stability: Revisit

We have \( A \beta = \text{Loop Gain} = \frac{A}{1/\beta} \)

\[ \therefore 20 \log |A\beta| = 20 \log A - 20 \log (1/\beta) \]

If \( \phi < -180^\circ \) Stable