Non-Linear Schrödinger Equation
The two wave equations of pulse evolution of light in an optical fiber which were introduced in the preceding discussion are:

\[ \nabla_\bot^2 F + \left\{ \varepsilon (\omega) K_0^2 - \tilde{\beta}^2 \right\} F = 0 \]  
(33.1)

\[ -2j\beta_0 \frac{\partial \tilde{A}}{\partial z} + (\tilde{\beta}^2 - \beta_0^2) \tilde{A} = 0 \]  
(33.2)

The terms in the two equations retain their meanings. Since the frequency band of optical signal is narrow, the quantity \( \tilde{\beta} \) is very close to \( \beta_0 \) and so:

\[ (\tilde{\beta}^2 - \beta_0^2) \approx 2\beta_0(\tilde{\beta} - \beta_0) \]  
(33.3)

Substituting this approximation in equation (33.2) we obtain:

\[ \frac{\partial \tilde{A}}{\partial z} + j(\tilde{\beta} - \beta_0) \tilde{A} = 0 \]  
(33.4)

Since we are concerned with a finite band of frequencies, the approximation of equation (33.3) is not true everywhere and there is a small difference between the two. Applying a Taylor series expansion around \( \beta_0 \) we have:

\[ \beta(\omega) = \beta_0 + (\omega - \omega_0)\beta'(\omega_0) + \frac{(\omega - \omega_0)^2}{2}\beta''(\omega_0) + \cdots \]  
(33.5)

If we define:

\[ \frac{\partial^n \beta(\omega)}{\partial \omega^n} (\text{at } \omega = \omega_0) \triangleq \beta^n(\omega_0) \triangleq \beta_n \]  
(33.6)

Then equation (33.5) can be re-written as:

\[ \beta(\omega) = \beta_0 + (\omega - \omega_0)\beta_1 + \frac{(\omega - \omega_0)^2}{2}\beta_2 + \cdots \]  
(33.7)

The dielectric constant \( \varepsilon \) of the optical fiber, which has both linear and non-linear terms, is given by the square of the refractive index 'n' added with a negligibly small value \( \Delta n \) due to the presence of non-linearity. The following approximation thus holds:

\[ \varepsilon = (n + \Delta n)^2 \approx n^2 + 2n\Delta n \]  
(33.8)

As discussed earlier, the quantity \( \Delta n \) can be expressed as a sum of the non-linear and the loss terms as shown below:

\[ \Delta n = n^2|\vec{E}| - j\frac{\alpha}{2K_0} \]  
(33.9)

The quantity \( \tilde{\beta} \), which is a function of \( \omega \), can be expressed as:

\[ \tilde{\beta}(\omega) = \beta(\omega) + \Delta \beta(\omega) \]  
(33.10)
The term $\Delta \beta(\omega)$ can be obtained from the following expression containing the field distribution function $F$.

$$
\Delta \beta(\omega) = K_0 \frac{n(\omega) \iint |\Delta n(\omega)| |F|^2 \, d\Omega}{\beta(\omega) \iint |F|^2 \, d\Omega}
$$

(33.11)

Here $d\Omega$ is the differential area of cross-section. Let us now write the expression for the inverse Fourier transform of the envelope function of the signal. This expression is given as:

$$
A(z, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}(z, \omega - \omega_0) e^{i(\omega - \omega_0)t} \, d\omega
$$

(33.12)

Substituting this expression in the equation (33.4) and solving, we obtain an equation in the time domain given as:

$$
\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - j\beta_2 \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A = -j\gamma |A|^2 A
$$

(33.13)

The quantity $\gamma$ is known as the non-linearity parameter and is related to the non-linearity coefficient $\gamma$ and the confinement of the light. It can be expressed as:

$$
\gamma = \frac{n_2 \omega_0}{c A_{\text{eff}}}
$$

(33.14)

the term $A_{\text{eff}}$ is the effective area of cross-section of the optical fiber within which the light is confined and mathematically, this area can be expressed as:

$$
A_{\text{eff}} = \left( \frac{\iint |F|^2 \, d\Omega}{\iint |F|^4 \, d\Omega} \right)^2
$$

(33.15)

The quantity $F$ is the transverse field distribution inside the optical fiber and knowing $F$ enables us to determine the effective area $A_{\text{eff}}$ within which the light is confined in an optical fiber. This, in turn, determines the non-linearity parameter $\gamma$ from equation (33.14). Depending upon the core size of the optical fiber, the field distribution etc. the typical values of $A_{\text{eff}}$ range between 1-100µm². Correspondingly, the typical values for $\gamma$ range between 1-100 W⁻¹/Km.

The equation (33.13) gives the behaviour of evolution of the optical signal envelope with time as the signal propagates along the optical fiber. Let us know have a physical insight into the equation.

If the optical fiber is loss-less ($\alpha = 0$), linear ($\gamma = 0$) and the quantity $\beta_2$ is negligibly small, the resultant equation is given as:

$$
\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} = 0
$$

$$
\Rightarrow \frac{\partial z}{\partial t} = -\frac{1}{\beta_1}
$$

(33.16)
The quantity $\partial z/\partial t$ is the group velocity $v_g$ of the optical pulse inside the optical fiber and is correctly given by equation (33.16) because the term $\beta_1$ is, in fact, the inverse of the group velocity of the optical pulse inside the optical fiber (as can be derived from equation (33.6)). Thus for a linear, loss-less optical fiber the optical pulse evolution equation reduces to an equation expressing the group velocity of the pulse inside the optical fiber.

Let us now assume that, we move along with the optical pulse with the group velocity. In this case, the term $\partial A/\partial t$ goes to zero, because there is no relative motion between the observer and the pulse. This observation caters the need of defining a new time frame with respect to a moving observer, moving at the group velocity. This new time $T$ can be expressed as:

$$T = t - \frac{z}{v_g} = \frac{1}{\beta_1} z, \quad \beta_1 = \frac{1}{v_g}$$

Thus,

$$\beta_1 \frac{\partial A}{\partial t} = \beta_1 \frac{\partial A}{\partial T} = \beta_1 (1 - \beta_1 v_g) = 0$$

Substituting this in equation (33.13), we obtain:

$$\frac{\partial A}{\partial z} - j \frac{\beta_2}{2} \frac{\partial^2 A}{\partial T^2} + \frac{\alpha}{2} A = -j \gamma |A|^2 A$$

Equation (33.19) represents the Non-Linear Schrödinger equation (NLS). Thus the NLS is the approximated differential equation governing the behaviour of the pulse evolution of an optical pulse inside an optical fiber in presence of non-linear effects. Before going into the solution of the NLS, let us first have a comprehensive physical understanding of the NLS as we did in the preceding case.

Let us assume that, $\beta_2$ is negligibly small and the fiber is linear i.e. $\gamma = 0$. The NLS, hence, reduces to:

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2} A = 0$$

The above equation is a simple linear differential equation whose solution is:

$$A(z) = A(0) e^{-\frac{\alpha z}{2}}$$

$\alpha$ being the attenuation constant, the above equation suggests that the optical pulse launched into the optical fiber would undergo an exponential decay w.r.t distance with an attenuation constant $\alpha$. This situation is similar to any general lossy optical medium in which the launched light decays exponentially. So, the term $\frac{\alpha}{2} A$ represents the loss in an optical fiber. The terms on the RHS of equation (33.19) is the result of the presence of non-linearity in the optical fiber material and the quantity $\gamma$ is (as already discussed) the non-linear parameter. The second term on the LHS of equation (33.19) is proportional to $\beta_2$ which is, in fact, the rate of change of group velocity as a function frequency and signifies that different frequencies travel with different group velocities. This phenomenon of different wavelengths travelling with different velocities is known as dispersion. So, the
second term in the LHS of equation (33.19) basically represents dispersion in the optical fiber.

In order to solve the NLS numerically, we define two operators as shown below:

\[
\text{Dispersion Operator: } \hat{D} \equiv j \frac{\beta_2}{2} \frac{\partial^2}{\partial T^2} \\
\text{Non - Linearity Operator: } \hat{N} \equiv -j\gamma|A|^2
\]

(33.22)  
(33.23)

One may note the fact that due to the presence of the differential operator, the dispersion operator eases the solution of the equation in the frequency domain because a differentiation in the time domain transforms to a frequency multiplication in the Fourier transform domain. Similarly, the non-linear operator eases the solution of the NLS in the time domain because of the presence of the envelope function A which is a time domain function. These two notions suggest a very important fact that the NLS can be solved relatively easily if it is solved simultaneously in the frequency as well as in time domains. However, since both the non-linear and dispersion effects are inherently weak effects, during the solution of the NLS for one the other effects may be assumed to be absent. Considering one effect at a time also enables us to have a better physical insight to the pulse evolution behaviour due to the individual effects.

Under the above assumptions, the pulse evolution is studied using the Split-Fourier Step method. In this method the medium under study is subdivided into smaller sections each having equal length ‘\(\Delta z\)’. The length chosen is such that the dispersion and the non-linear effects remain weak effects and the consideration of one effect at a time is adequate. Each subsection is then divided into two equal halves and the operator \(\hat{D}\) is applied in the frequency domain until the spectrum reaches the end of the half. At this point the time domain expression is obtained and the \(\hat{N}\) operator is applied to obtain the pulse envelope at that point. Then for the next half, again the Fourier transform is taken and the \(\hat{D}\) operator is applied until the spectrum reaches the end of the first subsection where the \(\hat{N}\) is applied to the time domain expression. This trend is repeated until the entire medium is spanned and the pulse evolution over the entire optical fiber can, thus, be obtained. The figure below may help the reader to understand better:

![Figure 33.1: Split Fourier Steps](image)

Although analytical solutions of the NLS exist for certain specific pulse shapes, complex pulse shapes can be solved numerically using the above method. Let us first assume a Gaussian pulse with a pulse power of \(P\) and a standard deviation of \(T_0\). That is:
\[ A(T) = e^{-\frac{T^2}{2T_0^2}} \]  

(33.24)

The characteristic dispersion length ‘\( L_D \)’ over which the dispersive effects become significant for the Gaussian pulse can be expressed as:

\[ L_D = \frac{T_0^2}{|\beta_2|} \]  

(33.25)

The characteristic non-linearity length ‘\( L_{NL} \)’ over which the non-linear effects become significant for the Gaussian pulse can be expressed as:

\[ L_{NL} = \frac{1}{\gamma P} \]  

(33.26)

Once the characteristic lengths are defined, four possible cases of the solution of the NLS arise which are based on the actual physical length if the optical fiber under consideration. If we consider a section of an optical fiber with a length \( L \) (and for simplicity of comprehension let \( \alpha = 0 \)), then the four possible cases are stated below:

**Case 1:** \( L \ll L_D, L \ll L_{NL} \)

In this case the chosen length of the optical fiber behaves just like a simple medium to transport light from the input to the output without any significant modification occurring in the pulse characteristics.

**Case 2:** \( L \gg L_D, L \ll L_{NL} \)

This scenario occurs in case of a low power, very narrow optical pulse. As the pulse power is low, the non-linear effects are negligible and the resultant modification that occurs to the pulse is the pulse broadening due to dispersion which has been already discussed under linear optics. This situation is also known as the group velocity limited (GVD) regime and most often this is the scenario that takes place in optical communication.

**Case 3:** \( L \ll L_D, L \gg L_{NL} \)

This situation occurs in case of a sufficiently broad pulse with high power so that non-linear effects are significant. In this case, a phenomenon known as the self-phase-modulation (SPM) is observed due to variation of the refractive index of the material by the pulse shape itself. SPM shall be discussed in detail later. This regime is called as the Non-Linearity limited regime.

**Case 4:** \( L \gg L_D, L \gg L_{NL} \)

This case arises when the pulse width is very narrow but there is high pulse power. In this scenario, both dispersion as well as non-linear effects co-exists and we obtain a very special kind of pulse propagation known as solitonic propagation.