Lecture 8
Relating $\phi, \psi$
and the filters
Generic Analysis Filter Bank
(Two-band, because dyadic)
Two band Analysis Filter Bank:

In the Haar MRA:

Crude lowpass

\[ \frac{1 + z^{-1}}{2} \]

\[ \frac{1 - z^{-1}}{2} \]

Crude highpass
Actual Haar Analysis

Filter Responses:

\[ \frac{1+z^{-1}}{2} \]

\[ \left| \cos \frac{\omega}{2} \right| \]

\[ \frac{1-z^{-1}}{2} \]

\[ \left| \sin \frac{\omega}{2} \right| \]
Ideal Analysis Filter Response:

Lowpass filter

LPF_{ID}

HPF_{ID}

Highpass filter
The Ideal Two Bank Filter Bank:

Input → LPF_{ID} → \downarrow 2 → \uparrow 2 → \text{Analysis} → HPF_{ID} → \uparrow 2 → \uparrow 2 → LPF_{ID} → + → \text{out}
Why are the ideal filters unattainable?
1. The ideal filters are infinitely noncausal.
   Frequency response → Impulse response
Inverse Discrete Time Fourier Transform

Ideal frequency response

$$H_{\text{ideal}}(\omega)$$

Impulse response = \text{Inverse DTFT of } H_{\text{ideal}}(\omega)
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{\text{ideal}}(w) \cdot E \cdot d\omega. \]
For the ideal lowpass filter this would be:

\[ h[n] = \begin{cases} 
\frac{\sin \frac{\pi n}{2}}{\pi n}, & n \neq 0 \\
\frac{1}{2}, & n = 0 
\end{cases} \]
Exercise:
Calculate the ideal impulse response of HPFID!
Disqualifications of the ideal filter:

1. Infinite noncausality!
Disqualification 2: the ideal filter is unstable!

$$\sum_{n} |h[n]| \text{ divergent}$$
Disqualification 3: The ideal filter is IRRATIONAL!!
Example of an irrational system function:

\[
\mathcal{Z}^{-1} \left\{ \frac{1}{z} \right\} = \text{ROC } \mathbb{C}, \quad \mathcal{Z}[n] = \frac{1}{n^0} \mu[n] \\
0^0 = 1, \quad n^0 = n(n-1)...1, \quad n \geq 1
\]
A realizable two-band filter bank:
provided

$H_0(z), \ G_0(z),$

$H_1(z), \ G_1(z)$ are

all rational system

functions!
$h_0(z)$ and $g_0(z)$ aspire to be ideal lowpass discrete filters with cutoff $\frac{\pi}{2}$.
$H_1(z)$ and $G_1(z)$ aspire to be highpass ideal filters with cutoff $\frac{\pi}{2}$!
Haar MRA:

\( \phi(t) \in V_0 \subset V_1 \)

\( \phi(t) \) should be expressible in terms of

\( \phi(2t - n) \quad \text{for } n \in \mathbb{Z} \)
$\phi(t)$

$\phi(2t)$  $\phi(2t-1)$
\[ \phi(t) = \phi(2t) + \phi(2t-1) \]

Dilation equation
Dilation equation

Coefficients for $\phi(t)$

\[ \uparrow \quad 1 \quad 1 \quad 1 \quad \\
\quad 0 \quad \]
\psi(t) \quad \text{Haar Wavelet}

\mathcal{V}_1 \quad \text{should be expressible in terms of basis } \{ \phi(2t-n) \}_{n \in \mathbb{Z}}
Graphically:

\[ \psi(t) \]

\[ \phi(2t) \]

\[ \phi(2t-1) \]
If \( g[n] \) = impulse response of the highpass filter in the two-band filter bank.
Coefficient sequence in this dilation equation

$$\psi(\tau) = \frac{1}{1 \uparrow 0} - 1$$

for $$\psi(\tau)$$
Dilation equation for $\psi(t)$:

$$\psi(t) = \phi(2t) - \phi(2t-1)$$
$h[n]$: impulse response of lowpass filter in two-band filter bank:

essentially: . . . .
\[ \phi(t) = \sum_{n \in \mathbb{Z}} h[n] \phi(2t-n) \]

\textit{dilation equation for } \phi(t)
\[ \psi(t) = \sum_{m \in \mathbb{Z}} g(m) \phi(2t - m) \]