LECTURE 4

WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING
1. Functions or signals as generalized vectors
2. Connections(s) between $L^2(\mathbb{R})$ functions and sequences.
2-Dimensional space corresponding to this paper:

\[ \mathbf{v}_1 = \mathbf{v} \cdot \mathbf{\hat{u}}_1 \]
\[ \mathbf{v}_2 = \mathbf{v} \cdot \mathbf{\hat{u}}_2 \]

\[ \mathbf{v} = \mathbf{v}_1 \mathbf{\hat{u}}_1 + \mathbf{v}_2 \mathbf{\hat{u}}_2 \]
\[ \begin{align*}
\epsilon & = \frac{2L}{R_1 + R_2} \\
\frac{1}{2} V & = k_1 \hat{u}_1 \\
\frac{1}{2} V & = k_2 \hat{u}_2 \\
\frac{1}{2} V & = \hat{u}_1 + \hat{u}_2 \\
V & = \frac{1}{2} (\hat{u}_1 + \hat{u}_2) \\
\text{Parallelogram Law}
\end{align*} \]
\[ \hat{v} = \kappa_1 \hat{u}_1 + \kappa_2 \hat{u}_2 \]
An infinite (countably infinite) dimensional vector is a sequence 

\[ x[n], \quad n \in \mathbb{Z} \]

set of integers

\[ \uparrow \text{index} \]
Sequence: a vector each n: different 'dimension' of the vector
'Dot product' of vectors

\[ \hat{v}_2 \rightarrow \hat{u}_1 \]

\[ e_1 = e_{11} \hat{u}_1 + e_{12} \hat{u}_2 \]

\[ e_2 = e_{21} \hat{u}_1 + e_{22} \hat{u}_2 \]
\[ \frac{1}{2} \cdot \frac{1}{2} = e_{11} e_{21} + e_{12} e_{22} \]

Sum of products of corresponding coordinates.
$N$-dim vectors

$\overrightarrow{e_1}: e_{11} \ e_{12} \ldots \ e_{1N}$

$\overrightarrow{e_2}: e_{21} \ e_{22} \ldots \ e_{2N}$

$\overrightarrow{e_1} \cdot \overrightarrow{e_2} = \sum_{k=1}^{N} e_{1k} e_{2k}$
$x_1[n], x_2[n], n \in \mathbb{Z}$

"Dot product" or inner product

$\langle x_1, x_2 \rangle$

Assume real sequences now.
\[ \langle x_1, x_2 \rangle = \sum_{m=-\infty}^{+\infty} x_1^*(m) x_2(m) \]

We would like to define the squared norm of \( x \) to be \( \langle x \rangle \).
What do we want of a 'norm'?

'Vector' $x$; essentially a sequence $x[n], n \in \mathbb{N}$.
'Norm' of sequence $x$

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$$

should be

and further:
\[ \| x \| \geq 0 \]

and \[ \| x \| = 0 \]

\[ \iff \quad x = 0 \cdot \]

\[ \text{i.e.} \quad x[n] = 0 \quad \forall n \in \mathbb{Z} \]
If $x_1, x_2$ are real

$$\langle x_1, x_2 \rangle = \sum_{n=-\infty}^{+\infty} x_1(n) x_2(n)$$
\[ \langle x, x \rangle = \lim_{n \to \infty} x[n]^2 \]

As long as \( x[n] \) real \( \forall n \in \mathbb{Z} \), this satisfies the requirements of 'norm'.
The following small change for complex sequences: $\langle x_1, x_2 \rangle$

$$= \sum_{n=-\infty}^{+\infty} x_1[m] \overline{x_2[m]}$$

**Complex Conjugate**
\[ \langle x_1, x_2 \rangle = \overline{\langle x_2, x_1 \rangle} \]

(i) Complex conjugate
(ii) linear in first argument

\[
\langle a_1 x_1 + a_2 x_2 \rangle x_3
= a_1 \langle x_1, x_3 \rangle + a_2 \langle x_2, x_3 \rangle
\]
(iii) Mononegativity (positive definite)

\[ \langle x, x \rangle \geq 0 \quad \text{and} \quad x = 0 \iff \langle x, x \rangle = 0 \]
Standard inner product:

\[ \langle x_1, x_2 \rangle = \sum_{m=-\infty}^{+\infty} x_1[m] x_2[m] \]
Exercise:
Verify the properties of

1) Conjugate Commutativity
\[ \langle x_1, x_2 \rangle = \langle x_2, x_1 \rangle \]

2) Linearity in first argument
(iii) Positive definiteness:

\[
\langle x, x \rangle = \lim_{m \to \infty} \sum_{n=-\infty}^{\infty} x[m] \overline{x[n]}
\]

\[
= \lim_{m \to \infty} \sum_{n=-\infty}^{\infty} |x[n]|^2 = 0 \quad \text{iff} \quad \forall n \quad x[n] = 0
\]
Extension to uncountably infinite dimension:

every \( t, t \in \mathbb{R} \)
is a different dimension
\[ x(t), \quad t \in R \]
\[ x(t) \text{ for particular } t \]
\[ \text{is the } \ell^t \text{ coordinate} \]
"Dot product" or "inner product" between two functions

\[ \langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y(t) \, dt \]
Exercise: Verify the properties of
(i) Conjugate commutativity
(ii) Linearity in first argument
(iii) Positive definiteness
Parseval's theorem

\[ x(t) \xrightarrow{\text{ } } \hat{x}(\nu) \]

\[ \hat{x}(\nu) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi\nu t} \, dt \]

Frequency in Hz
\[ x(\Omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t} dt \]

Angular frequency variable for continuous time

\[ \Xi = 2\pi f \]
Parseval's Theorem:

\[ x(t) \xrightarrow{\text{Fourier Transform}} \hat{x}(\omega) \]

\[ y(t) \xrightarrow{\text{Fourier Transform}} \hat{y}(\omega) \]
The inner product in time

\[ \langle x, y \rangle_{+\infty} = \int_{-\infty}^{\infty} x(t) y(t) dt \]

is Equal
to the inner product in frequency \( \nu \):
\[
\langle \hat{x}, \hat{y} \rangle = \int_{-\infty}^{\infty} \hat{x}(\nu) \hat{y}(\nu) d\nu
\]
\( \vec{r} \) is independent of coordinate system
\[ x(t) = \sum_{v} x(v) e^{\frac{t}{\tau}} dv \]

Reconstruct \( x(t) \) from its components, \( x(v) \).
\[ \phi(t) \]

\[ \cdots + c_i \phi(t+i) - c_j \phi(t-j) + c_0 \phi(t) + q \phi(t-1) + \cdots \]