LECTURE 34

CONSTRUCTING THE LATTICE AND ITS VARIANTS.
One 'module':

\[ H_m(\mathbb{Z}) \rightarrow H_{m+1}(\mathbb{Z}) \]

\[ \tilde{H}_m(\mathbb{Z}) \rightarrow H_{m+1}(\mathbb{Z}) \]
Given:
\[ H_n(z) = (-1)^n Z H_m \left( \frac{z}{2m-1} \right) \]
...we have
\[ H_{m+1}^2 (z) = - \frac{1}{z} H_m (-\frac{1}{z}) \]

Conjugate Quadrature relation is carried
Construction of lattice structure:

Go the other way:

\[ \frac{H_{m+1}}{H_m} \]
Inductive (recursive) lattice relations:

\[ H_m \rightarrow H_{m+1} \]
\[
H_{m+1}(z) = \frac{-2}{2} H_m(z) + K_{m+1} \rho H_m(z)
\]

\[
A_{m+1}(z) = \frac{-2}{2} H_m(z) - K_{m+1} H_m(z).
\]
Consider:

\[ H_{m+1}(\mathbb{Z}) \sim K_{m+1} H_{m+1}(\mathbb{Z}) \]
Equal to:

$$H_m(z) + K^2 H_m(z)$$

$$+ K_m H_{m+1}(z)$$

$$- K_{m+1} H_m(z)$$
\[ = \left( 1 + k^2 \kappa_{m+1} \right) H_m(c(z)) \]

(eliminated)

\[ H_m(z) \]
\[ H_m (z) = \frac{H_m (z) - K_{m+1} H_{m+1} (z)}{1 + K_{m+1}^2} \]
Obtain $k_{m+1}$? Look at forward recursion!
\[ H_{m+1}(z) = \text{length } 2(\text{m+1}) \]

\[ H(z) + k_{m+1} \leq H(z) \]

\[ \text{length } 2m \]

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We shall show inductively:

\[
\text{Coeff of } z^0 \text{ in } H_m(z) = 1.
\]
Basis step: true:

System functions:

\[ 1 + K_1 z^{-1} H_1(z) - K_1 + z^{-1} \quad (\text{true}) \]
Let it be true for $H_m(z), m \geq 1$.

$$H_{m+1}(z) = H_m(z) - 2^{m-2} + K_{m+1} \geq H_m(z)$$
This recursive step carries the term from $H_m(z)$ to $H_{m+1}(z)$.
\[
H_m(z) = -\frac{(2m-1)}{z} H_m(-z) \\
H_m(z) = \sum_{k=0}^{\infty} \binom{m}{k} \frac{(-1)^k}{z^{k+1}} \left( \frac{z}{1 + z^2} \right)^k \\
\]
\[ H_m(z) = -h_{2m-1} + \ldots + \frac{(m)}{2} h_2 z^2 + \frac{(m)(2m-1)}{2^2} z^2 + \ldots \]

\[ \text{important} \]
As a consequence of

\[ H_{m+1}(z) = H_m(z) \]

\[ + K_{m+1} \sum_{m=1}^{\infty} \frac{H_m(z)}{z^{2m+1}} \]

Contributes \( \frac{1}{z^{2m+1}} \) term.
Coeff of $z^{(2m+1)}$
(highest reg power of $z$)

$= K_{m+1}$
Once we know $K_{m+1}$, we can 'peel off' one module:
$H_{m+1}$: known

$\hat{H}_{m+1}$: can be constructed
\[
H_{m+1}(z) - K_{m+1} \tilde{H}_{m+1}(z) \sim \frac{1 + K_{m+1}^2}{1 + K_{m+1}}
gives \ H_m(z).
\]
Example:

Length 4 filter:

\[ 1 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} \]
Problem: Obtain $k_2, k_1$. 
Given the length 4 filter

\[ 1 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} \]

This is \( k_{20} \).
\[ H_2(z) = 1 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} \]

\[ H_2(z) = \frac{-3}{z} H_2(-z) \]

\[ = \frac{-3}{z} \left\{ 1 - h_1 z + h_2 z^2 - h_3 z^3 \right\} \]
\[ H_2(z) = -h_3 z^2 + h_2 z - h_1 z + z \]

Consider numerator of "notional" \( H_1(z) \).
\[ H_1(z) = \frac{H_2(z) - k_2 H_2(z)}{1 + k_2^2} \]
\[ H_2(Z) = k_2 \frac{H_2(Z)}{h^2_3} \]

\[ 1 + h_1 Z + h_2 Z^2 + h_3 Z^3 \]

\[ + h_1^2 Z^3 + h_2^2 Z^4 + h_3^2 Z^5 \]

\[ + h_1 h_2 Z^4 + h_1 h_3 Z^4 - h_2 h_3 Z^5 \]

\[ \text{because: } 0 \]
\[ h_1 \perp h_2 \perp h_3 \]

means:

\[ h_2 \perp h_3 ; h_1 = 0 \]
Remaining:

\[
\frac{(1 + h_3^2)}{\left( h_1 - h_2 h_3 \right)^2} = \frac{1}{z}
\]
The length has reduced by 2.
\[ H_1(z) = \frac{1}{(1 + h_3)^2 + (h_1 - h_2 h_3) z^2} \]
\[ K = 1 + \frac{h_1 - h_2 h_3}{1 + h_2^2 / h_3^2} \geq -1 \]
The backward recursion

\[ H_{m+1}(z) = \frac{H_m(z) - K_{m+1} \tilde{H}_m(z)}{1 + K_{m+1}^2} \]
\( \theta \) is the coefficient of higher negative power in \( \frac{1}{\sin \theta} \).
2) As long as this coefficient is real, the denominator

\(1 + K^2 (1 + K_{\text{MTD}})\) poses no problem!
In the numerator $H_{m+1}(\mathcal{E}) - K_{m+1} H_{m+1}(\mathcal{E})$, there is a length reduction of $\sqrt{2}$. 
Reduction by 1 is easy:

it occurs by cancellation of \(-1\),

highest power of \(2\).
Additional reduction in degree by 1:
attributed to orthogonality and translates
Synthesis variant essentially the transpose of the analysis lattice.
Length 2 synthesis:
Transpose of:
i.e.: 

\[ -K_1 \quad \text{and} \quad K_1 \]
Inductive Analysis model:
Inductive Synthesis
module:

\[ \mathcal{K}_m + 1 \]