Lecture 32

‘Noble’ Identities

and the Haar Wavepacket Transform
'Noble' Identities

Theme: To deal with cascades of sampling rate changes and filters.
Example:
'Noble' Identity for Downsampling

\[ \sqrt{2} \xrightarrow{} H(z) \xrightarrow{} \] How to interchange
\[ x_1[n] = x[2m] \]
\[ \forall m \in \mathbb{Z} \]
\[ y[n] = (x_1 * h)[n] \]
\[ y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k] \]
\[ y[n] = \sum_{k=-\infty}^{\infty} x[2k] h[n-k] \]

How does this fit into the diagram?
Let us use:

\[ y[n] = \lim_{\ell \to \infty} \sum_{\ell = -\infty}^{\ell = \ell} h[l] x_1[n-l] \]
\[ x_1[n-k] = x[2n-2k] \]

\[ y[n] = \sum_{l=-\infty}^{+\infty} h[l] x[2n-2l] \]
Just before the 'equivalent' downsampler

\[ \downarrow 2 \]

Here,
we have:

\[
\sum_{l=-\infty}^{+\infty} h[l] x[n-2l]
\]

This looks like a convolution.....
the $h[l]$ seems to be located at the $2l$th place; all $(2l+1)$th places are 0.
Define $h_1[n] = 0$

$n$ odd

$n$ not a multiple of 2

$= h[n/2], n$ even
\[ y[n] = \sum_{l=-\infty}^{+\infty} h[l] x[2n-l] \]

'n' before \( \downarrow 2 \)
Conclusion:  
\[ \sqrt{2} \quad \rightarrow \quad H(z) \quad h[n] \quad \rightarrow \]  
is equivalent to ---
Impulse response of filter preceding downsampler
-- is the original impulse response upsampled by 2

\[ h[m] \rightarrow \uparrow 2 \rightarrow h[2m] \]
is efficient computationally.
is inefficient computationally!
move towards bringing together
Noble Identity for Upsampling:
Transposition in multivariate systems means the following: --
1. Reverse direction of signal flow

2. When reverting an (up-) (resp. down-) sampler
put there a corresponding (down-) resp. up-
Sampler of same factor.
Noble Identity for Downsampling
Corresponding transpose:
This is the 'mobe' identity for upsampling!
Exercise:
Prove the noble identity for upsampling $\uparrow 2$
Challenge!
Prove that transposition leads to a valid alternate structure.
Exercise:
Prove the more general 'noble' identities:
$H(Z)$

M: any positive integer
Haar filter bank

\[ \begin{align*}
1 + z^{-1} & \quad \rightarrow \quad \sqrt{2} \\
1 - z^{-1} & \quad \rightarrow \quad \sqrt{2} \\
\end{align*} \]

repeat \( B \)

\( B' \)
There are 4 Cascade structures distinctly:
Using 'noble' identity for $\sqrt{2}$
4 cascade structures become:
\[ V_2 \rightarrow V_1 \rightarrow V_0 \]
\[
1 + \bar{z} \quad 1 + \bar{z}^2
\]
\[
(1 + \bar{z}) (1 + \bar{z}^2)
\]
\[
= 1 + \bar{z} + \bar{z}^2 + \bar{z}^3
\]
The sequence
1 1 1 1
↑ expands
0 \phi(t) in
terms of \phi(4t-k)
Indeed: \[ \phi(t) = 0 \times 4 + 2 \times 3 + 1 \]

\[ \phi(4t) + \phi(4t-1) + \phi(4t-2) + \phi(4t-3) \]
Haar Wavelet (expected)
and its integer translates
Finally, for Will:

\[
\begin{align*}
\text{0} & \quad \text{1} \\
\text{1/4} & \quad \text{1/2} \\
\text{3/4} & \quad \text{1}
\end{align*}
\]

and its translates.