LECTURE 3

The Haar (MRA) Multiresolution Analysis
Wavelet translates at max resolution
We are essentially "peeling off" shell by shell using different dilates and translates!
Focal point:
Piecewise constant-approximation on unit intervals
What function $\phi(t)$ is such that its integer translates can span this space?
\( \mathbb{Z} \): set of integers

\( \forall \): 'for all'

linear space of functions: a set of functions, such that their linear combinations fall in the same set.
The space of piecewise constant functions on the standard unit intervals! \( \forall n \in \mathbb{N}, n+1 \subset \mathbb{Z} \)
\[ V_0 : \{ x(t), \text{ such that } \]
\[ x(\cdot) \in L^2(\mathbb{R}) \]
\[ \text{belongs to} \]
\[ \text{and } x(\cdot) \text{ is piecewise constant on all } \mathbb{N}, n+1 \in \mathbb{Z} \text{ (integer) } \]
We say $V_0$, because of piecewise constancy on intervals of size $2^n = 1$. 
Similarly

\[ V_1 : \{ x(t), x \in L^2(\mathbb{R}) \} \]

and \( x(t) \) is piecewise constant on standard \( 2^{-1} \) intervals \( J_n := [n, (n+1)2^{-1}] \) \( n \in \mathbb{Z} \) \( \frac{3}{2} \).
In general \( V_m = \{ x(t), \ x \in L^2_2(\mathbb{R}) \} \)
and \( x(\cdot) \) is piecewise constant on all
\( \{ n \cdot 2^{-m}, (n+1) \cdot 2^{-m} \} \) for \( n \in \mathbb{Z} \).
Example of \( x(\cdot) \in \frac{1}{2} \cdot \{ \mathbb{Z} \} \)
The absolute squared sum of the piecewise constant values, in all \( V_m \), must be convergent \( (E \in L_2(\mathbb{R})) \).
Example of $x(\cdot) \in \mathbb{V}_{-1}$

\[
\frac{1}{2^{(-1)}} = 2^1 = 2
\]
A ladder of subspaces'' implies:

\[ V < V_1 < V_0 < V_1 < V_2 \ldots \]

Intuitively $\rightarrow$ towards $L^2(\mathbb{R})$!
What happens when we go leftwards?

$\cdots V < V_1 < V_0 < V_1 C V_2 \cdots$

| -2 | -1 | 0 |

Piecwise constants on larger intervals?
$\ell_2$ norm of functions going leftward

$$= \sum_{n} \frac{1}{m} \left| C_n \right|^2$$

$$= \left| \text{m} \right| \sum_{n} \left| C_n \right|^2 \xrightarrow{m \to \infty} -\infty$$
If \[ 2^{m_1} \leq \left| \sum_{n} c_m \right|^2 \text{ must converge, no matter how large } |m_1| \text{ is,} \]
then \[ \sum_{n} |c_m|^2 \rightarrow 0 \]
Moving upwards:

Union

\[
\bigcup_{m \in \mathbb{Z}} V_m = \mathcal{L}_2(\mathbb{R})
\]
$L^2(R)$

closure
cover up patches

$U$ $V$ $W$
meets all
covers. interior
Maring 'downwards'

Intersection!

\[ \bigcap_{m \in \mathbb{Z}} V_m = \{0\} \]

trivial subspace
We say \( \{f_1, \ldots, f_k \ldots \} \) span a linear space if any function in the space can be generated by linear combinations of this set.
What function \( \phi(t) \) and its integer translates span \( V_0 \)?
Any function in $V_0$ can be written as:

$$\sum_{n \in \mathbb{Z}} \phi(t-n)$$

piecewise constant translates of $\phi$.}
... + 0.2 \phi(t+1) + 0.7 \phi(t) + 1.5 \phi(t-1) + 1.3 \phi(t-2) + ...
Any space $V_m$ can be similarly constructed

$$V_m = \text{span} \{ \phi(2^t - n) \}$$

$n \in \mathbb{Z}$
$\phi(t)$ is called the Scaling Function (in this case of the Haar Multiresolution Analysis!)
\[ V \supset V_1 \supset V_0 \supset V_1 \supset V_2 \supset \ldots \]

-2 -1

With these properties, it is called a Multiresolution Analysis (MRA).
Axioms of a
Multiresolution
Analysis

Ladder of subspaces of $L^2(\mathbb{R})$: $\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots$
Such that
\[
\overline{U \bigvee_{m \in \mathbb{Z}} V_m} = \mathcal{L}_2(\mathbb{R})
\]

1.

2.

\[ \bigcap_{m \in \mathbb{Z}} V_m = \{0\} \]

Contd...
3. There exists \( \phi(t) \) such that

\[ V_0 = \text{Span}\{ \phi(t-n) \} \quad \forall n \in \mathbb{Z} \]

4. \( \{ \phi(t-n) \} \) is an orthogonal set!
5. If $f(t) \in V_m$, then $f(2^m t) \in V_0$ and $m \in \mathbb{Z}$.

6. If $f(t) \in V_0$, then $f(t - n) \in V_0$.
Theorem: Given these axioms, there exists $\psi(\cdot) \in L^2(\mathbb{R})$ so that

$\{\psi(2^m t - n)\}_{m \in \mathbb{Z}, n \in \mathbb{Z}} \subseteq \text{span} L^2(\mathbb{R})$