LECTURE 28

JPEG 2000 5/3 FILTER BANK AND SPLINE MRA
\[ \phi_0(t) = \phi_0(2t) + \phi_0(2t-1) \]

\[ \phi_1(t) = \frac{1}{2} \phi_1(2t) + \phi(2t-1) + \frac{1}{2} \phi(2t-2) \]
\[ \frac{1}{2} (1 + \xi')^2 \]

\( \Phi_1(t) \) is piecewise linear.
'Splines' are polynomial pieces or piecewise polynomial interpolants.
'Shortcoming' of $\Phi(t)$: not orthogonal to all integer translates
orthogonal to $\Phi(t)$
Let without loss of gen
\[ g_{x(-2)} = (1 + x_1 x_2)^2 \]
Alias cancellation:

\[ \Phi(z) H_0(-z) + g_1(z) H_1(-z) = 0. \]

Replace \( z \leftarrow -z \).
\[ H_0(x) \chi_0(-x) + H_1(x) \chi_1(-x) = 0 \]
Perfect reconstruction condition

\[ G_0(z) H_0(z) + G_1(z) H_1(z) \overset{D}{\rightarrow} G \]
Once perfect reconstruction conditions satisfied in a 2-band filter bank
the analysis and synthesis filters can be exchanged to give another PR 2-band filter bank
\[ \sigma_0(z) = (1 + \frac{1}{\pi z^2}) \]

Alias cancellation:

\[ \sigma_0(z) H_0(-z) + \sigma_1(z) H_1(-z) = 0 \]
\[ \frac{G_0(z)}{G_1(z)} = -\frac{H_1(-\bar{z})}{H_0(-\bar{z})} \]

One simple choice
num = num
den = den
\[ g_0(x) = -H_1(-x) \]
\[ g_1(x) = H_0(-x). \]

Here:
\[ -H_1(-x) = \left(1 + \frac{1}{x^2}\right) \]
\[ H_1(z) = -\{ (1 - z)^{-1} \}^2 \]
\[
\text{\textgreater} \quad g_{0}(\zeta)H_0(\zeta) + g_{1}(\zeta)H_1(\zeta) = c_0 z^{1-A}
\]
\[ g_0(x) H_0(x) \]
\[ + H_0(-x) H_1(x) \]
\[ = C_0 \]
\[ H_1(x) = -g_0(-x) \]
$2\mathrm{CO} = \mathrm{C}_2\mathrm{O} + \mathrm{O}_2$
\[ \mathcal{G}_0(\varpi) \mathcal{H}_0(\varpi) = K_0(\varpi) \]

\[ K_0(\varpi) - K_0(-\varpi) \rightarrow \mathcal{C}_0 \varpi \]
1. Kill even samples in Im z - transform of \( K_0(z) \)
2. Out of the remaining odd samples, only one is nonzero. \[
\frac{D^{th} \text{ sample} = C_0.}
\]
We shall choose $H_0(z)$ also to have 2 zeros at $z = -1$.
\( H_0(2) \text{ to have a factor} \left(1 + \frac{1}{\xi^2}\right)^2 \)
Parsimoniously ("stingily") extending $H_0(z)$ to retain impulse response symmetry.
introduce a factor

\((1 + h'_0 \bar{z}' + \bar{z})^{1-1} \frac{-1}{-2}\)

only one degree of freedom
\[ H_0(z) = \left( 1 + \frac{z}{1 + h_0 z^2 + z^2} \right)^2 \]

Let us now consider \( G_0 H_0 \).
\[ G_0(z) H_0(z) \]
\[ = \left( 1 + \frac{-1}{z} \right) ^2 \left( 1 + \frac{-1}{z} \right) ^2 \]
\[ \left( 1 + h_0 \frac{-1}{z^2} + \frac{z}{z} \right) \]
\[(1 + \frac{1}{z})^4 = 1 + 4z + 6z^2 + 4z^3 + z^4\]
Convolved with $h_0$
If we make
\[ 4 + h_0 = 0 \]
\[ h_0 = -4 \]
we have retained only ONE ODD SAMPLE!
Therefore we choose
\[ h_0 = -4 \]
\[ H_0(2) = \frac{-1}{2\sqrt{1 + 2(1 - 4z^2 + z)}} \]
$H_0(z) =$

\[
\begin{pmatrix}
1 & 2 & 1 \\
1 & * & 1 \\
0 & \uparrow & 0 \\
\end{pmatrix}
\begin{pmatrix}
-4 \\
1 \\
1 \\
\end{pmatrix}
\]
\[ H_0(z) = \frac{1}{1 - 2z - 6z^2 - 2z^3 - 3z^4} \]

Length = 5
$S_0(\tau) = \left(1 + \tau \frac{z}{\tau} \right)^{-1} \frac{z^2}{\tau} - 2$

$= 1 + 2 \tau + \tau^2$

$\text{length} = 3$
This is the celebrated 5/3 filter bank in JPEG-2000.
573 filter bank

Lengths:

Antithero

Ordinary
\[ G_0(z) = -H_1(-z) \]

\[ H_1(z) = -G_0(-z) \]

\[ = -\left(1 - \frac{1}{z^2}\right)^2 \]
\[
\begin{align*}
Q_1(z) &= H_z(-z) \\
&= 1 + az^2 - \frac{b}{2}z^4 + \frac{c}{3}z^6 + \frac{d}{4}z^8 \\
&= 1 + 2z^2 - 6z^4 + 4z^6 + 2z^8 + z^{10}
\end{align*}
\]
\[ \frac{1}{x} = c_1 \hat{u}_1 + c_2 \hat{u}_2 \]
Bipartite orthogonal basis to $\ell_1$ $\ell_2$ $\ell_1$ $\ell_2$ $\ell_2$ $\ell_1$