LECTURE 25

THE THEOREM OF

(DYADIC)

MULTIRESOLUTION

ANALYSIS
Filter banks with different analysis and synthesis wavelets & scaling functions: biorthogonal filter bank.
We are always talking about filter banks with perfect reconstruction.
Same analysis and synthesis
wavelets/scaling
functions: ORTHOGONAL
FILTER BANKS
In general

$k$th analysis branch

\[ \psi(a_0) \]

Input $x$, $a_0 > 1$, $k$: all integers
$k^{th}$ synthesis branch:

Output of $k^{th}$ analysis branch
\[ \hat{\psi}(\Omega) = \frac{\hat{\psi}(\Omega)}{SDS(\psi, a_0)(\Omega)} \]

SDS (sum of dilated spectra)
\[ S_D S(4, a_0)(52) \]
\[ = \sum_{k=-\infty}^{+\infty} |\hat{\psi}(a_0 k)|^2 \]
Provided, \( \exists \; C_1, C_2 \)

there exist

\( 0 < C_1 \leq SDS(\nu, \varphi)(n) \leq C_2 < \infty \)

\( \psi(\cdot) \) is admissible
Because of $G$, $\psi$ was meaningful!

$\tilde{\psi}$ was admissible,

Bounds on SDS: $\frac{1}{c_2}, \frac{1}{c_1}$. 
Challenge/exercise:

Come up with examples of $\varphi$ which satisfy the requirement with $C_1 = C_2$. 

Construction of an orthogonal filter bank:

Define:

\[ \hat{\psi}(s) = \frac{\psi(s)}{\sqrt{SDS(\psi, a_0)}(s)} \]
ψ is admissible.

Consider

\[ SD(\tilde{\psi}, a_0)(B) \]
\[\text{SDS}(\varphi, a_0)(n) = \sum_{k=-\infty}^{+\infty} \left| \varphi(a_0^n) \right|^2 \]
$$SDS(\psi, a_0)(n)$$

$$= SDS(\psi, a_0)(a_0^n)$$

Proof in general:
$$SDS(\psi, a_0)(a_0, n)$$

$$= \sum_{k = -\infty}^{\infty} \left| \hat{\psi}(a_0, a_0, n) \right|^2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{\psi}(a_0, a_0, n) \right|^2$$

Proved.
\[ SDS(\psi, a_0)(n) = \sum_{k=-\infty}^{+\infty} |\hat{\psi}(a_k n)|^2 \]

\[ SDS(\psi, a_0)(n) \equiv 1 \]

for all \( n \)
\[ \psi \text{ is admissible on account of satisfying upper = lower = 1 bound on } \text{SDS} \]
\(\psi\) can be used both on the analysis and synthesis side.
Dyadic Multiresolution Analysis (MRA): other MRA examples? Haar MRA, Daubechies’ MRAs and so on.
Special Case:

\[ a_0 = 2 \]

Wavelet obeys the requirement \((a_0 = 2)\),

\(0 < c_1 \leq \text{SDS} \leq c_2 \leq \infty\).
and the wavelet admits discretizing the translation parameter
On the kth analysis branch: output is broadly a BANDPASS FUNCTION.
Band occupancy = \pi

We could use a sampling rate = \frac{21\pi}{\pi} = 2
If you simply use the Nyquist criterion, \( f_s = 2 \) sampling freq.

\[ 2\pi f_s = 4\pi, \quad f_s = 2 \]
But we can also do with a 

sampling rate 

$2\pi f_s = 2 \times \pi = 2\pi 

\implies f_s = 1$. 
Suppose we do use a sampling rate = 1

Aliases: shifts of original spectrum by $\pi/1 \times K$ for integer $K$. 
$4\pi$ forward  $4\pi$ back

$-4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$

unaffected!
Let us now focus on $a_0 = 2$.

We need to use a logarithmic change of sampling.
On the $k^{th}$ branch, sampling rate relates to $2^k$. 
Axioms of a (dyadic) MRA:

(i) Ladder axiom

\[ V_{-2} < V_{-1} < V_0 < V_1 < V_2 \ldots \]
(ii) \( \overline{V V_m} = L_2(\mathbb{R}) \sum_{m \in \mathbb{Z}} \)

(iii) \( \Pi V_m = \{0\} \sum_{m \in \mathbb{Z}} \)
(iv) If \( x(t) \in V_0 \)
\[ x^{(m)}(2t) \in V_m \]

(Implicitly this provides for logarithmic sampling.)
(V) Translation

\[ x(t) \in V_0 \]

\[ x(t-n) \in V_0 \quad \forall \ n \in \mathbb{Z} \]
(vi) orthogonal basis

\[ \exists \phi(t) \text{ so that } \{ \phi(t-n) \} \text{ basis for } n \in \mathbb{Z} \neq V_0. \]
Theorem of Dyadic MRA:

Given axioms

(i) \rightarrow (vi), \exists a \alpha,
\Psi(x) \in V_1; \Psi(t) \in L_2(R),
Such that \( \{ \psi(2^n t - n) \} \) forms an orthogonal basis for \( L^2(\mathbb{R}) \)