LECTURE 20

THE TIME FREQUENCY PLANE AND ITS TILINGS
Time bandwidth product

Time variance

Frequency variance

$(t^2) \times \text{Frequency variance}$
Time bandwidth product

\[ t \cdot \Omega \geq 0.25 \quad \text{for any} \quad x \in L^2(\mathbb{R}) \]
\[ x(t) = e^{-\frac{t^2}{2}} \]

is an example of an optimal function.
More general optimal function

\[ e^{t^2/2} + 60e^{Re(60)} \text{ negative} \]
LSI System -> Composite LSI System
Impulse response of composite system

= Convolution of

\[ \delta(t-T) \]
Triangular pulse

2T
Time bandwidth product of \( x(t) \) with \( t \geq 0 \) for \(-1 \leq t \leq 1\), \(1-|t|\) for \(0 \leq |t| \leq 1\), and \(-1\) for \(-1 \leq t \leq 0\).
Time variance: \[ x(t) \text{ centre} = 0 \]

\[ \frac{\| t x(t) \|_2^2}{\| x(t) \|_2^2} \]
\[ \|x(t)\|_2^2 = 2 \int_0^1 (1 - t)^2 \, dt \]
\[ \lambda = 1 - t \]
\[ \| x(t) \|_2^2 = 2 \int_0^1 x^2 \, dt \]
\[ \|t \times (t)\|_2^2 \]

\[ = 2 \int_0^1 t^2 (1-t)^2 \, dt \]

using symmetry
\[
\begin{align*}
&= 2 \int_{0}^{1} t(1-2t+t^2) \, dt \\
&= 2 \int_{0}^{1} \left( t - 2t^2 + t^4 \right) \, dt
\end{align*}
\]
\[ = 2 \left\{ \frac{t^3}{3} - 2 \cdot \frac{t^4}{4} + \frac{t^5}{5} \right\} \bigg|_0^1 \]

\[ = 2 \left\{ \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right\} \]
= \ 2 \cdot \left\{ \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right\}

= \ 2 \cdot \frac{10 - 15 + 6}{2 \times 15} = \frac{1}{15}
Time variance

\[ \frac{1}{15} \frac{213}{213} = \frac{1}{15} \times \frac{3}{2} \]

\[ = 0.1 \]
Frequency variance

\[
= \left\| \frac{dx(t)}{dt} \right\|^2_2 \frac{2}{\|x(t)\|^2_2}
\]
\[ \left\| \frac{dx(t)}{dt} \right\|_2^2 = 1^2 + 1^2 = 2 \]
\[ \|x\|_2^2 = \frac{2}{3} \]

Frequency variance = \[\frac{2}{2\sqrt{3}}\]

= 3
Time bandwidth product

\[
\begin{align*}
\text{Time} & \times \text{Frequency Variance} \\
= & \ 0.1 \times 3 \\
= & \ 0.3
\end{align*}
\]
Exercise: Evaluate the overall impulse response.
Exercise 2:
Obtain the time-bandwidth product of the impulse response.
Fourier duality: \( \hat{A} \rightarrow \left( \frac{\sin A \hat{r}}{BF} \right)^2 \)

\( A, B \) constants
As a consequence of Fourier duality.
For the function \( (\frac{\sin At}{Bt})^2 \), time variance = frequency variance of \( \sqrt{\frac{1}{2}} \).
Frequency \cdot Variance = \text{time variance of } \Lambda

\Rightarrow \text{Time bandwidth product} = 0.3
The time bandwidth product is invariant to Fourier transformation.
From this example, we see it is possible to have two functions, one compactly supported and one not, with the same $\Delta^2 \delta_n$.
"Time-Frequency Plane"
"Occupancy" of \( x(t) \in L^2(\mathbb{R}) \) in this time-frequency plane
(to \( \sigma_0 \))

Centre

\( 20 \sqrt{t} \)

Time-Frequency plane
Uncertainty principle: Rectangle area cannot be smaller than \(20 \times 2\sqrt{2} = 40 \times \sqrt{2} = 4 \times 0.5 \geq 4 \sqrt{0.25}\)
"Tiling" the time frequency plane: Covering this plane with rectangular "tiles" corresponding to such functions.
Take any other function to be analyzed: \( y(t) \)
"Tool" function

\[ F(t) = x(t) \]

\[ \int_{-\infty}^{\infty} y(t) x^2(t) \, dt \]
From Parseval's theorem:

\[ \int_{-\infty}^{\infty} Y(m) \overline{Y(n)} \, dp_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(n) \overline{Y(m)} \, dp_2 \]
"Chirp" function

\[ \text{Sim} \left[ \frac{2\pi t}{\text{frequency}} \right] \left[ \frac{t}{\text{instantaneous frequency}} \right] \]
Frequency $\uparrow$ Constant

$\rightarrow$ Time

$\bar{\gamma}(t) =$ Constant