LECTURE 11
TWO CHANNEL FILTER BANK
Upansampler:

\[ x_{\text{in}}[m] \rightarrow \uparrow 2 \rightarrow x_{\text{out}}[m] \]

\[ X_{\text{out}}(Z) = X_{\text{in}}(Z^2) \]
Downsampled:

\[ x_{in}[m] \xrightarrow{\sqrt{2}} x_{out,D} \]

\[ X_{out,D}(z) = \frac{1}{2} \left\{ X_{in}(z) + X_{in}(-z) \right\} \]
$\frac{1}{2}(1^n + (-1)^n)$
\[ Y_1(z) = H_0(z) \times (z) \]
\[ Y_2(z) = H_1(z) \times (z) \]
\[ \frac{1}{2} \left\{ 1^n + (-1)^n \right\} \]
\[ Y_5(z) = \frac{1}{2} \left\{ Y_1(z) + Y_1(-z) \right\} \]
\[ Y_b(z) = \frac{1}{2} \left\{ Y_2(z) + Y_2(-z) \right\} \]
\[ Y_7(z) = Y_5(z) G_0(z) \]
\[ Y_8(z) = Y_6(z) G_1(z) \]
\[ Y(z) = Y_7(z) + Y_8(z) \]

\[ = \text{(let us consider } Y_8 \text{ first)} \]
\[ Y_8 (z) = Y_6 (z) g_1 (z) \]

\[ Y_6 (z) = \frac{1}{2} \left\{ Y_2 (z) \right\} \]

\[ Y_6 (-z) \]
\[ y_2(z) = x(z) H(z) \]

\[ y_2(\bar{z}) = x(\bar{z}) H_1(\bar{z}) \]
In total

\[ Y(z) = \frac{J_0(z)X(z) + J_1(z)X(-z)}{1} \]
\[ J_0(z) = \frac{1}{2} \left\{ G_0(z) H_0(z) + G_1(z) H_1(z) \right\} \]
\[
J_1(z) = \frac{1}{2} \left\{ G_0(z) H_0 + G_1(z) H_1(z) \right\}
\]
\[ Y(z) = \text{(in domain)} \]

linear combination

of \( x(z) \) and \( x(-z) \)
What does $X(-z)$ do?

$z \leftarrow e^{jw}$
\(( -z ) \quad = \quad e^{j\pi} \quad e^{jw} \quad \text{for} \quad z = e^{jw}\)
Replacing $e^{jw}$ by $e^{j(w \pm \pi)}$
Example spectrum: $x(e^{j\omega})$. 

-1, -\pi/2, 0, \pi/2, \pi

$\omega$
Actually:

Periodic repeat: $x(t)$

(period $2\pi$) $\rightarrow \infty$
\[ x(\text{csc}(\omega \pm \pi)) : \]

Principal interval
Expanding the principal interval: $x(w + \pi)$
$X(-z)$ term or equivalently $X(-e^{jw})$ ...
or \( X(e^{j(\omega \pm \pi)}) \)

term is the consequence of aliasing
Aliasing does two things:

(i) one frequency manifested as another

(ii) increasing actual frequency → decreasing apparent
If we want $Y(Z)$ to reconstruct $X(Z)$, we must first do away with aliasing!
i.e. \( J_1(z) = 0 \)

\[ q_0(z) H_6(-z) + q_1(z) H_1(-z) = 0 \]
\[
\frac{g_1(z)}{g_0(z)} = -\frac{H_0(-z)}{H_1(-z)}.
\]
Very simple choice:

\[ G_1(x) = \pm H_0(-x) \]

\[ G_0(x) = \mp H_1(-x) \]
More generally:

\[ g_1(z) = \pm R(z) H_0(-z) \]

\[ g_0(z) = \mp R(z) H_1(-z) \]

\( R(z) \): the factor "cancelled"
Interpret

\[ g_1(z) = + H_0(-z) \]

Ideally:

\[ 1 \quad H_e(j\omega) \]

\[ \begin{array}{cccc}
-\pi & -\pi/2 & \pi/2 & \pi \\
\end{array} \]
\[ H_0(2z) \mid \omega \leftrightarrow z \leftrightarrow e^{j\omega t} + \pi \]

= \[ H_0(e^{j\omega t} \pm \pi) \]
"Zoomed" \( H_0(e^{j\omega}) \)

![Diagram with frequency axis labeled and intervals marked from \(-3\pi\) to \(\pi\), with values \(-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}\) indicated.]
Showing only principal interval

Ideal filter

\( H(e^{j(\omega \pm \pi)}) \)
\( g_1(z) = H_0(-z) \)

makes a lot of sense for the ideal filter!
Exercise: Show, similarly that ideal $\text{HPF, } \frac{11}{2}$ maps to $\frac{L \ell}{\sqrt{7}}$.