1 Uncertainty Product

There is a bound on simultaneous time and frequency localization. So essentially one cannot localize as much as one wants simultaneously in time and frequency.

One can define for time centered function \( x(t) \), time variance

\[
\sigma^2_t(x) = \frac{\|tx(t)\|^2_2}{\|x(t)\|^2_2}
\]

Recall meaning of time center, time center

\[
t_0 = \int \frac{t|x(t)|^2}{\|x(t)\|^2_2} dt
\]

By time centered, we mean \( t_0 = 0 \).

Similarly for frequency, frequency center or frequency mean

\[
\Omega_0 = \int \frac{\Omega|\hat{x}(\Omega)|^2}{\|\hat{x}(\Omega)\|^2_2} d\Omega
\]

By frequency centered, we mean \( \Omega_0 = 0 \).

Analogous to time variance, for frequency centered function \( \hat{x}(\Omega) \) frequency variance

\[
\sigma^2_\Omega(x) = \frac{\|\Omega\hat{x}(\Omega)\|^2_2}{\|\hat{x}(\Omega)\|^2_2}
\]

If function is not time centered and frequency centered then one need to take second moment around respective centers.

For an \( L_2(\mathbb{R}) \) function the uncertainty product i.e. product of time and frequency variance is lower bounded by 0.25.

\[
\sigma^2_t(x).\sigma^2_\Omega(x) \geq \frac{1}{4}
\]

**Example 1.** Calculate uncertainty product of \( e^{-|t|} \) for all \( t \).

**Sol.** First we need to verify if the function is \( L_2(\mathbb{R}) \) and check if it is centered in time and frequency.

\[
\|x(t)\|^2_2 = \int_{-\infty}^{+\infty} |x(t)|^2 dt = 2 \int_{0}^{+\infty} e^{-2t} dt
\]
\[ \|x(t)\|_2^2 = 1 \]

From the sketch of \( x(t) \) one can clearly say that it is symmetric about \( t = 0 \) and is a real and even function of \( t \). Since it real and even function in time domain it should have real and even fourier transform too. So obviously the function \( x(t) \) is time and frequency centered.

![Sketch of x(t)](image)

Time variance \( \sigma_t^2 \):

\[
\sigma_t^2(x) = \frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} = 2 \int_{0}^{\infty} t^2 e^{-2t} dt = \frac{1}{2}
\]

Frequency variance \( \sigma_\Omega^2 \):

\[
\sigma_\Omega^2(x) = \frac{\|j \Omega \hat{x}(\Omega)\|_2^2}{\|\hat{x}(\Omega)\|_2^2}
\]

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\]

By applying Parseval’s Theorem it becomes

\[
\sigma_\Omega^2(x) = \left\| \frac{dx(t)}{dt} \right\|_2^2
\]

Now

\[ x(t) = e^{-|t|} \]

So

\[
\left\| \frac{dx(t)}{dt} \right\|_2 = \|x(t)\|_2
\]

\[ \sigma_\Omega^2(x) = 1 \]

Uncertainty Product:

\[
\sigma_t^2(x) \cdot \sigma_\Omega^2(x) = \frac{1}{2} \cdot 1
\]

\[ \sigma_t^2(x) \cdot \sigma_\Omega^2(x) = 0.5 > 0.25 \]

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Example 2. Calculate uncertainty product of raised cosine function.
Sol.

\[
x(t) = 1 + \cos t \quad -\pi < t < \pi \\
= 0 \quad \text{otherwise}
\]

From the above figure it is clear that the function \( x(t) \) is real and even, so it is already time and frequency centered. Since the function is real and even, hence

\[
\|x(t)\|_2^2 = \int_{-\infty}^{+\infty} |x(t)|^2 dt \\
= 2 \int_0^{+\infty} |x(t)|^2 dt \\
= 2 \int_0^\pi (1 + \cos t)^2 dt \\
= 2 \int_0^\pi (1 + \cos^2 t + 2 \cos t) dt
\]

Consider

\[
\int (1 + \cos^2 t + 2 \cos t) dt = \int (1 + \frac{1 + \cos 2t}{2} + 2 \cos t) dt
\]

Now let us sketch \( \cos t \) and \( \cos 2t \)

From the above figure it is clear that they have zero integral over \([0, \pi]\)

So the above integral becomes

\[
\|x(t)\|_2^2 = 2 \int_0^\pi (1 + \frac{1}{2}) dt = 3\pi
\]
Frequency variance $\sigma_\Omega^2$:

$$\sigma_\Omega^2(x) = \left\| \frac{dx(t)}{dt} \right\|_2^2 \left\| x(t) \right\|_2^2$$

Now

$$\frac{d}{dt}x(t) = \frac{d}{dt}(1 + \cos t) = -\sin t$$

$$\left\| \frac{dx(t)}{dt} \right\|_2^2 = 2 \int_0^\pi \sin^2 t dt = \int_0^\pi (1 - \cos 2t) dt = \frac{\pi}{2}$$

So frequency variance $\sigma_\Omega^2 = \frac{1}{3}$

Time variance $\sigma_t^2$:

We need

$$\int t^2|x(t)|^2 dt = \int t^2(1 + \cos t)^2 dt = \int t^2 \left( 1 + \frac{1 + \cos 2t}{2} + 2 \cos t \right) dt$$

Let us consider the term

$$\int t^2 \cos mt dt$$

Now our limit is 0 to $\pi$, therefore we do not need to look at the ‘sin’ term which are zero at $t = 0$ and $t = \pi$. Again we do need terms containing ‘t’ at $t = 0$. We need only consider the term $2t \frac{\cos mt}{m^2} \bigg|^\pi_0$

For $m=1$, $2t \cos t \bigg|^\pi_0 = 2\pi \cos \pi = -2\pi$

For $m=2$, $2t \frac{\cos t}{4} \bigg|^\pi_0 = 2\pi \frac{\cos 2\pi}{4} = 0.5\pi$

Putting these values in above equation

$$\int_0^\pi t^2 \left( 1 + \frac{1 + \cos 2t}{2} + 2 \cos t \right) dt = \int_0^\pi 1.5 t^2 dt + \int_0^\pi t^2 \frac{\cos 2t}{2} dt + 2 \int_0^\pi t^2 \cos t dt$$

$$= \frac{\pi^3}{2} + \frac{\pi}{4} - 4\pi$$

Time variance $\sigma_t^2$

$$\sigma_t^2 = \frac{2 \int_0^\pi t^2|x(t)|^2 dt}{2 \int_0^\pi (1 + \cos t)^2 dt}$$
\[ \sigma_t^2 = \frac{\pi^2}{3} - \frac{5}{2} \]

Uncertainty Product:

\[ \sigma_t^2(x) \sigma_{\Omega}^2(x) = \left( \frac{\pi^2}{3} - \frac{5}{2} \right) \times \frac{1}{3} \]

\[ \sigma_t^2(x) \sigma_{\Omega}^2(x) = \frac{\pi^2}{9} - \frac{5}{6} (> 0.25) \]