1 Introduction

We continue in this lecture to build upon the particular class of filter bank which we have introduced in the previous lecture called a Conjugate Quadrature Filter (CQF) bank.

2 Conjugate Quadrature Filter bank

For the perfect reconstruction system we must first do away aliasing. The alias cancellation equation for the two band filter bank is given by

\[ G_0(Z)H_0(-Z) + G_1(Z)H_1(-Z) = 0 \]

\[ \frac{G_1(Z)}{G_0(Z)} = - \frac{H_0(-Z)}{H_1(-Z)} \]

Equating the numerator and denominator we get the relation between \( G_0(Z) \), \( H_1(-Z) \), \( G_1(Z) \) and \( H_0(-Z) \) as

\[ G_1(Z) = -H_0(-Z) \]
\[ G_0(Z) = H_1(-Z) \]

The relation between the analysis HPF (high pass filter) and analysis LPF (low pass filter) called a conjugate quadrature relationship, is given by

\[ H_1(Z) = z^{-D}H_0(-Z^{-1}) \]

Here \( z^{-D} \) term is used to introduce causality. Putting \( Z = e^{j\omega} \) in the above equation we get the frequency response equation as

\[ H_1(e^{j\omega}) = e^{-j\omega D}H_0(-e^{-j\omega}) \]

The magnitude response is given by

\[ |H_1(e^{j\omega})| = |e^{-j\omega D}H_0(-e^{-j\omega})| \]
\[ |H_1(e^{j\omega})| = |e^{-j\omega D}||H_0(-e^{-j\omega})| \]
\[ |H_1(e^{j\omega})| = |H_0(-e^{-j\omega})| \]

\( H_0(Z) \) is a Low pass filter with a real impulse response (real coefficients), therefore
The magnitude response of LPF $H_0(Z)$ is symmetric along the magnitude axis and phase response is anti-symmetric along the frequency axis $\omega$.

$$H_0(-e^{-j\omega}) = H_0(e^{-j(\omega+\pi)})$$

**NOTE:** LPF with cutoff frequency $\frac{\pi}{2}$ (With shift by $\pi$ on $\omega$) $\Leftrightarrow$ HPF with cutoff frequency $\frac{\pi}{2}$

We have shown,

$$H_1(Z) = z^{-D}H_0(-Z^{-1})$$

For the perfect reconstruction the equation must satisfy,

$$G_0(Z)H_0(Z) + G_1(Z)H_1(Z) = C_0z^{-D}$$

$$H_1(-Z)H_0(Z) - H_0(-Z)H_1(Z) = C_0z^{-D}$$

$$(-1)^{-D}z^{-D}H_0(Z^{-1})H_0(Z) - H_0(-Z)z^{-D}H_0(-Z^{-1}) = C_0z^{-D}$$

We need the following for perfect reconstruction systems,

$$(-1)^{-D}H_0(Z^{-1})H_0(Z) - H_0(-Z)H_0(-Z^{-1}) = C_0$$

If we consider the Haar filter then the relationship between $H_0(Z)$ and $H_1(Z)$ is given by,

$$H_0(Z) = 1 + z^{-1}$$

$$H_0(-Z^{-1}) = 1 - z$$

The above equation is non-causal so to make it causal by inserting delay, we get the below equation,

$$z^{-D}H_0(-Z^{-1}) = z^{-D}(1 - z)$$

Here $z^{-D}$ retains causality.

If $D$ is odd,

$$H_0(Z)H_0(Z^{-1}) + H_0(-Z)H_0(-Z^{-1}) = -C_0$$

$$H_0(Z)H_0(Z^{-1}) + H_0(-Z)H_0(-Z^{-1}) = \text{Constant}$$

Putting $Z = e^{j\omega}$, we get the above equation in the frequency domain as,

$$H_0(e^{j\omega})H_0(e^{-j\omega}) + H_0(-e^{j\omega})H_0(-e^{-j\omega}) = \text{Constant}$$

For real impulse response we have,

$$H_0(e^{-j\omega}) = \overline{H_0(e^{j\omega})}$$

$$H_0(e^{j\omega})\overline{H_0(e^{j\omega})} + H_0(e^{j(\omega+\pi)})\overline{H_0(e^{j(\omega+\pi)})} = \text{Constant}$$

$$|H_0(e^{j\omega})|^2 + |H_0(e^{j(\omega+\pi)})|^2 = \text{Constant}$$
Above equation is called the power complementary equation. For perfect reconstruction system,
\[ H_0(Z)H_0(Z^{-1}) + H_0(-Z)H_0(-Z^{-1}) = \text{Constant} \]

Lets assume \( \kappa_0(Z) = H_0(Z)H_0(Z^{-1}) \)
\[ \kappa_0(Z) + \kappa_0(-Z) = \text{Constant} \]

We are going to choose even length of \( H_0(Z) \), i.e. \( D \rightarrow \text{Odd} \)

\[ h[n] : \quad h_0 \ h_1 \ h_2 \ \ldots \ h_D \]

Similarly, \( H_0(Z^{-1}) \) is given by,
\[ h[n] : \quad h_D \ \ldots \ h_2 \ h_1 \ h_0 \]

Here \( H_0(Z)H_0(Z^{-1}) \) corresponds to their convolution in time domain
\[ (h_0 \ h_1 \ h_2 \ \ldots \ h_D) \bigotimes (h_D \ \ldots \ h_2 \ h_1 \ h_0) \]

Let impulse response \( h[k] \) be as given below
\[ h[k] : \quad h_0 \ h_1 \ h_2 \ \ldots \ h_D \]

And impulse response \( g[k] \) is given below which is mirror image of \( h[k] \), that means \( g[k] = h[-k] \)

\[ g[k] : \quad h_D \ \ldots \ h_2 \ h_1 \ h_0 \]

Similarly \( g[n - k] \) is shown below

\[ g[n - k] : \quad h_0 \ h_1 \ h_2 \ \ldots \ h_D \]

The convolution between \( h[k] \) and \( g[k] \) is given
\[ \kappa_0[n] = \sum_{k=-\infty}^{k=+\infty} h[k]g[n - k] \]

Here \( h[k] \) is causal and filter length is \((D + 1)\).
The convolution at the sample \( n \) is \( g[n] \).
Shown below is the multiplication of $h[k]$ and $g[k]$ (which is shifted by $n$ samples).

\[
k_0[n]: \begin{array}{cccccccc}
h_0 & h_1 & h_2 & \ldots & h_n & h_{n+1} & \ldots & h_D \\
\uparrow & \uparrow & 0 & & & & \uparrow & n
\end{array}
\]

In $Z$-domain $\kappa_0(Z) = H_0(Z)H_0(Z^{-1})$.

The $m^{th}$ sample of the filter $k_0[m]$ is $<h[k], h[k \pm m]>$

Let $m = 2$ and filter length $4$ ($D = 3$)

\[
k_0[2]: \begin{array}{cccc}
h_0 & h_1 & h_2 & h_3 \\
\uparrow & \uparrow & \uparrow & 0 \quad 2
\end{array}
\]

$k_0[2] = h_0h_2 + h_1h_3$

If $m = -2$ and filter length $4$ ($D = 3$)

\[
k_0[-2]: \begin{array}{cccc}
h_0 & h_1 & h_2 & h_3 \\
\uparrow & \uparrow & \uparrow \quad -2 \quad 0
\end{array}
\]

$k_0[-2] = h_0h_2 + h_1h_3$

That means the convolution between $h[n]$ and $h[-n]$ is symmetrical.

\[
\kappa_0(Z) + \kappa_0(-Z) = \text{Constant}
\]

\[
\frac{1}{2}\{\kappa_0(Z) + \kappa_0(-Z)\} = \text{Constant}
\]

From the above equation the summation $\frac{1}{2}\{\kappa_0(Z) + \kappa_0(-Z)\}$ represents the nonzero sample value at even location and zero sample value at the odd location.

Let $\kappa_0(Z)$ correspond to the sequence $k_0[n]$, $\frac{1}{2}\{\kappa_0(Z) + \kappa_0(-Z)\}$ impulse response is shown below.
But from the equation \( \frac{1}{2} \{ \kappa_0(Z) + \kappa_0(-Z) \} = \text{Constant} \), we want the non-zero sample value only at zero location and zero sample value for odd and even location.

So at the even location \( m = 2l \) and \( m \neq 0 \) and \( (l \in \mathbb{Z}) \), we want zero sample value.

Let Daubechies filter with length 4 \((D = 3)\)

\[
h_0[n] : \quad h_0 \quad h_1 \quad h_2 \ldots h_3
\]

In the Haar case, \((1 - z^{-1})\) represents a High pass filter.

Here we consider the Daubechies filter with length 4 so two \((1 - z^{-1})\) in the High pass filter which means \((1 - z^{-1})^2\) factor in HPF.

Similarly, low pass filter has a factor \((1 - z^{-1})^2\).

A Daubechies low pass filter with length 4 is given by

\[
H_0(Z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3}
\]

We can write this equation in the factor of \((1 - z^{-1})^2\) i.e.

\[
H_0(Z) = (1 + z^{-1})^2(1 + B_0 z^{-1})
\]

In the above equation, we need three zeros.

Two zeros are already chosen at unit circle which are \(-1, -1\) and one zero is selected based on value of \(B_0\). This value can be obtained by comparing the above two equations.

Expanding the above two equations

\[
\begin{align*}
H_0(Z) &= (1 + 2z^{-1} + z^{-2})(1 + B_0 z^{-1}) \\
H_0(Z) &= 1 + (2 + B_0)z^{-1} + (1 + 2B_0)z^{-2} + B_0 z^{-3}
\end{align*}
\]

The dot product of the impulse response of LPF with its even shifts must be zero. We will use this constraint to find the value of \(B_0\) in the next lecture.