**1 Generalized vectors:**

A vector quantity or vector, provides the magnitude as well as the direction of a specific quantity.

**Example:** When giving directions to a point, it is not enough to say that it is \( x \) miles away, but the direction of those \( x \) miles must also be provided for the information to be useful. (Note that physical quantities are represented by Scalars, such as temperature, volume and time etc.)

Given a coordinate system in three dimensions, a vector may thus be represented by an ordered set of three components which represent its projections \( v_1, v_2, v_3 \) on the three coordinate axes.

\[
v = [v_1, v_2, v_3]
\]

The three most commonly used coordinate systems are rectangular, cylindrical, and spherical. Alternatively, a vector may be represented by the sum of the magnitudes of its projections on three mutually perpendicular axes:

\[
\vec{v} = v_1 \hat{u}_1 + v_2 \hat{u}_2 + v_3 \hat{u}_3
\]

The \( n \)-dimensional coordinate systems based on the Euclidean space (Cartesian space or \( n \)-space) represented by \( R^n \) or \( E^n \), under \( n \)-dimensions and \( n \)-vectors. Usually, the Euclidean space is formed by \( (X_1, X_2, X_3, ..., X_n) \) where \( n \) is equal to 8.
Parallelogram law of vector:
Let us take an example, in this using parallelogram law we can get the resultant vector. The resultant vector can be calculated as:

\[ \vec{v} = \vec{v}_1 + \vec{v}_2 \]

where \( \vec{v}_1 = k_1 \hat{u}_1 \)
and \( \vec{v}_2 = k_2 \hat{u}_2 \)
then \( \vec{v} = k_1 \hat{u}_1 + k_2 \hat{u}_2 \)

![Figure 2: Parallelogram law of vectors](image)

### 2 Relationship between functions, sequences, vectors:

One can intimately relate processing of a function to processing of equivalent sequence, and whatever we are doing to try and gain information from or modify a function can be done equivalently by processing or modifying that sequence corresponding to function. A sequence is like a vector and each \( n \) is a different dimension of that vector.

An infinite (countably infinite) dimension vector is a sequence \( x[n], n \in \mathbb{Z} \), where \( n \) is index and \( \mathbb{Z} \) is set of integers.

Now, we would like to extend other ideas of vectors to this context of infinite dimension vector.

**Dot product of vectors:**

Let,
\[ \vec{e}_1 = e_{11} \hat{u}_1 + e_{12} \hat{u}_2 \]
and \( \vec{e}_2 = e_{21} \hat{u}_1 + e_{22} \hat{u}_2 \) then dot product is \( \langle \vec{e}_1 \vec{e}_2 \rangle = e_{11}e_{21} + e_{12}e_{22} \). That means it is nothing but sum of products of corresponding coordinates.

Let two \( n \)-dimensional vectors as
\( \vec{e}_1 : e_{11}, e_{12}, ..., e_{1N} \) and \( \vec{e}_2 : e_{21}, e_{22}, ..., e_{2N} \) the dot product of these two vectors is
\[ \langle \vec{e}_1 \vec{e}_2 \rangle = \sum_{k=1}^{N} e_{1k}e_{2k} \]. These are also called as orthogonal coordinates.

Let two sequences, say \( x_1[n], x_2[n], n \in \mathbb{Z} \), the ‘dot product’ or ‘inner product’ is \( \langle x_1, x_2 \rangle \),
where
\[ \langle x_1, x_2 \rangle = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n] \]

In 2\text{D}, 3\text{D} space, we will calculate magnitude from the dot product, but in general n\text{D} space, we will use norm. Generally norm squared represents energy.

Let vector \( x \): essentially a sequence \( x[n], n \in \mathbb{Z} \), then the ‘norm’ of sequence \( x = \| x \| \) should be \( \langle x_1, x_2 \rangle^{1/2} \)

\( \| x \| \geq 0 \) and \( \| x \| = 0 \) iff \( x = 0 \) i.e. \( x[n] = 0 \) \( \forall \ n \in \mathbb{Z} \)

If \( x_1 \) and \( x_2 \) are real,

\[ \langle x_1, x_2 \rangle = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n] \]
\[ \langle x, x \rangle = \sum_{n=-\infty}^{+\infty} x^2[n] \]

As long as \( x[n] \) is real \( \forall n \in \mathbb{Z} \), this will satisfy norm requirements.

A small change will be applied for complex sequences as follows

\[ \langle x_1, x_2 \rangle = \sum_{n=-\infty}^{+\infty} x_1[n]\overline{x_2[n]} \]

Properties of Inner product:

1. Conjugate community:
\[ \langle x_1, x_2 \rangle = \overline{\langle x_2, x_1 \rangle} \]
\[ = \sum_{n=-\infty}^{+\infty} x_1[n]\overline{x_2[n]} \]
\[ = \sum_{n=-\infty}^{+\infty} x_2[n]\overline{x_1[n]} \]
\[ \langle x_1, x_2 \rangle = \overline{\langle x_2, x_1 \rangle} \]

2. Linear in first argument:
\[ \langle a_1 x_1 + a_2 x_2, x_3 \rangle = a_1 \langle x_1, x_3 \rangle + a_2 \langle x_2, x_3 \rangle \]
\[ = \sum_{n=-\infty}^{+\infty} (a_1 x_1 + a_2 x_2)x_3 \]
\[ = \sum_{n=-\infty}^{+\infty} a_1 x_1 x_3 + a_2 x_2 x_3 \]
\[ \langle a_1 x_1 + a_2 x_2, x_3 \rangle = a_1 \langle x_1, x_3 \rangle + a_2 \langle x_2, x_3 \rangle \]
3. Positive definite:

\[ \langle x, x \rangle = \sum x[n] \cdot x[n] \]
\[ \langle x, x \rangle = 0 \; \text{iff} \; x[n] = 0 \; \forall n \]

**Extension to uncountably infinite dimension:**
For any ‘\( t \)’, \( t \in \mathbb{R} \) is a different dimension and \( x(t), t \in \mathbb{R} \), means \( x(t) \) for a particular ‘\( t^{th} \)’ coordinate. Then the ‘dot product’ or ‘inner product’ between two functions \( x(t) \) and \( y(t) \) is

\[ \langle x, y \rangle = \int_{-\infty}^{+\infty} x(t)y(t)dt \]

**Parseval’s Theorem:**
The Parseval’s theorem states that the inner product of any two functions in time domain is equal to the inner product of those two functions in frequency domain.

Let \( x(t) \) be a function and \( \hat{x}(\nu) \) or \( \hat{x}(\Omega) \) is its fourier transform (in Hz or in radians) and defined as

\[ \hat{x}(\nu) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi\nu t}dt \; \text{or} \; \hat{x}(\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt \; \text{where} \; \Omega = 2\pi\nu \]

Let \( y(t) \) be a function and \( \hat{y}(\nu) \) or \( \hat{y}(\Omega) \) is its fourier transform (in Hz or in radians) and defined as

\[ \hat{y}(\nu) = \int_{-\infty}^{+\infty} y(t)e^{-j2\pi\nu t}dt \; \text{or} \; \hat{y}(\Omega) = \int_{-\infty}^{+\infty} y(t)e^{-j\Omega t}dt \; \text{where} \; \Omega = 2\pi\nu \]

The inner product of these in time domain is

\[ \langle x, y \rangle = \int_{-\infty}^{+\infty} x(t)y(t)dt \]

and it is equal to the inner product in frequency domain given by

\[ \langle \hat{x}, \hat{y} \rangle = \int_{-\infty}^{+\infty} \hat{x}(\nu)\overline{\hat{y}(\Omega)}d\Omega \]

That means \( \langle x, y \rangle = \langle \hat{x}, \hat{y} \rangle \)

The function \( x(t) \) can be reconstructed from its frequency components as

\[ x(t) = \int_{-\infty}^{+\infty} \hat{x}(\Omega)e^{-j\Omega t}d\Omega \]

**Applications of Parseval’s Theorem:**
The Parseval’s theorem is often used in many areas like physics and engineering etc, and it is written many of the times as

\[ \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |\hat{x}(\nu)|^2 d\nu \]

where \( \hat{x}(\nu) \) represents the continuous Fourier transform of \( x(t) \) and ‘\( \nu \)’ represents the frequency component of \( x \).
From this equation, the theorem tells that the total energy contained in a function \( x(t) \) over all time ‘\( t \)’ is equal to the total energy of the its Fourier Transform \( \hat{x}(\nu) \) over all frequency ‘\( \nu \)’.

For discrete time signals, the theorem becomes:

\[
\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |\hat{x}(e^{j\omega})|^2 d\omega
\]

where \( \hat{x}(e^{j\omega}) \) is the Discrete-Time Fourier transform (DTFT) of \( x \) and ‘\( \omega \)’ represents the angular frequency (in radians per sample) of \( x \).

For the Discrete Fourier transform (DFT), the relation becomes:

\[
\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\hat{x}[k]|^2
\]

where \( \hat{x}[k] \) is the DFT of \( x[n] \) and ‘\( N \)’ is length of sequence in both domain.

**Relation between continuous functions and sequences:**
Let \( x(t) \) be a continuous function and let \( \phi(t) \) be a unit step function in \([0, 1]\) interval, then \( x(t) \) can be written as

\[
x(t) = \ldots + C_{-1}\phi(t + 1) + C_0\phi(t) + C_1\phi(t - 1) + C_2\phi(t - 2) + \ldots
\]

It can be graphically represented as shown in figure 3.

![Figure 3: Relation between continuous functions and sequences](image.png)

Equivalence between continuous functions and sequences will be dealt in greater detail in subsequent lectures.