Here, we will consider some problems which can be analysed in terms of the basic elements of quantum mechanics.

**8.1 Tutorial 1: Some preliminaries**

1. In a photo-dissociation process, an incoming photon of frequency $\nu$ is absorbed by a positronium, and the electron is emitted at an angle $\theta$ with the direction of the incoming photon. What are the possible moments of the electron? Treat the electron and the positron non-relativistically.

2. Obtain the final frequency in Compton scattering by considering the electron non-relativistically. Carry out an expansion in inverse powers of $c$ to obtain the usual expression for Compton shift.

3. Carry out Planck’s analysis for radiation confined to a plane. What is the energy density per unit area, per unit wavelength? For what values of the wavelength is this a maximum? What is the total energy per unit area?

4. For a particle with charge $q$, moving in the presence of an external magnetic field, the canonical momentum is $mv - \frac{q}{2} \vec{r} \times \vec{B}$. Use Bohr’s quantization condition to obtain the energy levels of a particle with charge $q$ in the presence of $\vec{B}$.

**8.2 Tutorial 2: Elements of Quantum mechanics**

1. For a hydrogen atom with wave function

$$\psi = A r \sin \theta e^{-i\phi} e^{-r/2r_0}$$

normalise the wave function and consider the total current across the plane $x = 0$, $z = (0, \infty)$ $y = (0, \infty)$.

Obtain the wave function in the momentum space.

2. For a particle described by a wave function

$$\psi(x) = A x e^{-|x|/a}, \quad a > 0,$$

in a potential $V(x)$ which vanishes at infinity, obtain the energy and the potential, and calculate the average values of $|x|$, $1/|x|$. Obtain the wave function in the momentum space and calculate the average values of $p$ and $p^2$. Relate the average kinetic energy and the potential energy.

3. In the 3-dimensional vector space, write down a complete set of orthonormal basis vectors $a_i$, $i = 1, 2, 3$.

Show that they satisfy the closure property. In this basis, consider an operator $A$, with $Aa_1 = a_1 + a_2$, $Aa_2 = a_3$, $Aa_3 = 0$.

What are the eigenvalues and eigenvectors of $A$? Determine the eigenvalues and eigenvectors of $(A - A^+)/2$.

4. Normalize the 3-dimensional, particle wave function $1/(r^2 + a^2)$. Obtain the average values of $r$, $p^2$.

What is the probability that the particle is found in the region $? \quad$
5. Which of the operators $\vec{r}$, $\vec{p}$, $\vec{r} \cdot \vec{p}$, $\vec{r} \times \vec{p}$, $p^2$ are observables? Which of them commute with the Hamiltonian of a particle with charge $q$, in the presence of a magnetic field in the $z$ direction?

Write down the function which is the eigenfunction of $p^2$ and $\vec{p}$, and of $p^2$ and parity operator $\pi$.

8.3 Tutorial 3: Problems in 1-D

1. Obtain the wave function which has the minimum value for the product $\sigma_x \sigma_p$.

2. For a particle of mass $m$ in a potential

$$V(x) = -Z\delta(x), \quad Z > 0, \quad x < a, \quad a > 0$$

what is the minimum value of $Z$ for which a bound state exists? For what value of $Z$ is there a bound state with energy $E = -\hbar^2/2ma^2$? For this case, obtain the average value of $x$.

3. A particle of mass $m$ in a box with potential $V(x)$,

$$V(x) = 0 \quad \text{for} \quad 0 < x < a$$

$$= \infty \quad \text{for} \quad x < 0 \text{ or } x > a,$$

is described by the wave function

$$\psi(x, 0) = A \sin\left(\frac{(m + n)\pi x}{2a}\right) \cos\left(\frac{(n - m)\pi x}{2a}\right)$$

where $m$ and $n$ are integers. Obtain the average values of $H$ and $x$ as functions of time.

4. For a particle of mass $m$ described by the potential in

$$H = \frac{1}{2m}p^2 + \frac{1}{2}kx^2 + bx,$$

consider

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left[x + \frac{i}{m\omega}p + \frac{b}{k}\right],$$

and express $H$ in terms of $a$, $a^+$. Obtain the expressions for $[a, a^+]$, $[a, H]$, and eigenvalues and eigenfunctions of $a$.

5. Obtain the bound state energies and wave functions for a particle of mass $m$ in a potential

$$V(x) = -\frac{Z}{|x|} + \frac{a}{x^2}, \quad Z > 0, \quad a > 0.$$ 

Obtain the average value of $1/|x|$ for the ground state.

8.4 Tutorial 4: Problems in 2-D and 3-D

1. For a power-law potential,
obtain the dependence of the energy, its eigenfunctions, and the average values of $r$, on mass $m$, $Z$, $n$.

2. For a particle of mass $\mu$ in a two-dimensional potential

$$V(r) = -\frac{Z}{r} + \frac{c}{r^2},$$

obtain the lowest energy eigenvalue for a given angular momentum quantum number $m$, and the corresponding normalized wave function. Calculate the average values of $1/r$, $1/r^2$, and use the virial theorem to obtain the average kinetic energy for the state.

3. For a particle of mass $m$ described by the two-dimensional potential

$$V(r) = \frac{1}{2} kr^2 [3 + \cos(2\phi)] + br\cos(\phi),$$

use the creation and annihilation operators to write down the normalized wave function for the two lowest energy eigenstates. Calculate the average value of $y$ for the normalized state

$$\psi = \frac{1}{\sqrt{2}} (|\psi_0 > + |\psi_1 >)$$

as a function of time.

4. For a particle of mass $m$ in a 3-D potential

$$V(r) = \begin{cases} \infty & \text{for } r < R, \\ -V_0 & \text{for } R < r < R + a, \\ 0 & \text{for } r > R + a, \end{cases}$$

obtain an implicit expression for the bound state energies for the $l = 0$ states.

5. Obtain the wave functions and bound state energies for a 3-D potential

$$V(r) = \frac{1}{2} kr^2 + a/r^2.$$ For the lowest energy $l$ state, calculate the average values of $r^2$ and $1/r^2$. Use the virial theorem to obtain the average kinetic energy and verify the result by direct calculation.

6. An electron in a Coulomb potential is described by the wave function

$$\psi(\vec{r}, t) = A[f_0(t)e^{-r/r_0} + f_1(t)\frac{z}{r_0}e^{-r/2r_0}]$$

with $f_0(0) = f_1(0) = 1$. Determine the normalization constant $A$, and obtain the average values of $z$, $r$, $L_x$ and $L_z^2$ as functions of time.

7. For an electron in a strong magnetic field in the $z$ direction, evaluate the commutators $[H, \vec{L}]$, $[H, \vec{p}]$.

Which observables are constant in time?