In this module we consider the electron states in Quantum Dots under the influence of a magnetic field. The QDs may be realized by etching 2DEG hetero-structure, a destructive method. However it can also be obtained by using an electrostatic potential due to metal gates so as to confine the 2DEG electrons in certain regions. Such an arrangement, known as Split Gate, is schematically shown in Fig 7.1, which depicts a Split Gate made from GaAs/AlGaAs heterostructure with two dimensional electron gas below the surface; negative voltages applied to the gates deplete the 2DEG region to quantum dots indicated by its projection on top surface as circle.

**Fig 7.1: Schematic of a Split Gate**

Within the Effective Mass Approximation (EMA), the one electron wave function for non interacting electrons in the QD is,

\[ \Psi(\vec{r}, z) = \phi(\vec{r})\xi_i(z), \]

where \( \vec{r} \) is the position vector in the plane of the 2DEG (taken to be the x-y plane) and z-is the coordinate normal to the 2DEG; here \( \xi_i(z) \) is the eigen function corresponding to the \( i^{th} \) bound state (sub band) in the z-direction, characterizing the bulk 2DEG outside the QD, and \( \phi(\vec{r}) \) is the solution in the plane of the 2DEG satisfying the equation,

\[
\left[-\frac{\hbar^2}{2m^*} \nabla^2_{\vec{r}} + V_{eff}(\vec{r}) \right] \phi(\vec{r}) = E\phi(\vec{r}),
\]

where \( m^* \) is the effective mass in the plane of the dot, and the eigenvalue \( E \) and the eigenfunction \( \phi \) are to be determined on specifying the potential \( V_{eff}(\vec{r}) \). This potential is due to the space charge region under and around the split gates which produce the quantum dot, and has a quadratic dependence; as a first approximation, the potential in a quantum dot formed using a rectangular array of gates, is a parabolic of the form,

\[ V_{eff}(\vec{r}) = \frac{1}{2}m^*\omega_x^2x^2 + \frac{1}{2}m^*\omega_y^2y^2, \]

for which the bound states in the dot are the usual harmonic oscillator solutions, which can be found in any standard text book of Quantum Mechanics, with eigenvalues

\[ E(n_x, n_y) = \hbar\omega_x(n_x + 1/2) + \hbar\omega_y(n_y + 1/2), \]
where \( n_{x}, n_{y} = 0, 1, 2, \ldots \). For a AlGaAs/AGaAs split gate dot, these eigenvalues can be several meV in magnitude.

For further analysis, let us recapitulate some elementary quantum mechanics, and introduce the so-called raising (creation) and lowering (annihilation) operators, defined respectively as

\[
a_{x}^{\dagger} = \sqrt{\frac{m^{*} \omega_{x}}{2\hbar}} \left( x - i \frac{p_{x}}{m^{*} \omega_{x}} \right), \quad a_{x} = \sqrt{\frac{m^{*} \omega_{x}}{2\hbar}} \left( x + i \frac{p_{x}}{m^{*} \omega_{x}} \right),
\]

so that

\[
a_{x}^{\dagger} a_{x} = \frac{1}{\hbar \omega_{x}} \left( \frac{p_{x}^{2}}{2m^{*}} + \frac{1}{2} m^{*} \omega_{x}^{2} x^{2} \right) - \frac{1}{2},
\]

and similar entities corresponding to \( y \)-variables. Then the Hamiltonian for the problem can be written as

\[
\mathcal{H} = \hbar \omega_{x} (a_{x}^{\dagger} a_{x} + \frac{1}{2}) + \hbar \omega_{y} (a_{y}^{\dagger} a_{y} + \frac{1}{2}),
\]

so that \( a_{x}^{\dagger} a_{x} \) and \( a_{y}^{\dagger} a_{y} \) are number operators and \([a_{x}, a_{x}^{\dagger}] = 1 = [a_{y}, a_{y}^{\dagger}]\), where \([,]\) denotes a commutator.

Now note that for \( \omega_{x} = \omega_{y} = \omega_{o} \), one has degenerate eigen-states (in addition to spin degeneracy) due to radial symmetry of the problem:

\[
E_{n} = (n + 1) \hbar \omega_{o},
\]

where \( n = n_{x} + n_{y} = 0, 1, 2, \ldots \). The lowest state \( n_{x} = 0 = n_{y} \) is non degenerate; the next level \( n_{x} = 1, n_{y} = 0 \) and \( n_{x} = 0, n_{y} = 1 \) is doubly degenerate; and so on. In general, the \( n^{th} \) level \( E_{n} \) is \( (n + 1) \)-fold degenerate. These degeneracies correspond to different angular momentum states sharing the same energy.

To see this, recall that the angular momentum is

\[
\vec{L} = \vec{r} \times \vec{p}, \quad \text{and} \quad L_{z} = xp_{y} - yp_{x}.
\]

It is easy to show that

\[
L_{z} = i\hbar (a_{y}^{\dagger} a_{x} - a_{x}^{\dagger} a_{y}),
\]

using \([a_{x}^{\dagger}, a_{y}] = [a_{x}, a_{y}^{\dagger}] = [a_{x}^{\dagger}, a_{x}] = [a_{y}, a_{y}] = 0\).

It also follows that \([L_{z}, \mathcal{H}] = 0\), so that \( L_{z} \) is conserved.

Now define rotating creation and annihilation operators as

\[
a = \frac{1}{2}(a_{x} - ia_{y}), \quad b = \frac{1}{2}(a_{x} + ia_{y}),
\]

\[
a^{\dagger} = \frac{1}{2}(a_{x}^{\dagger} + ia_{y}^{\dagger}), \quad b^{\dagger} = \frac{1}{2}(a_{x}^{\dagger} - ia_{y}^{\dagger}).
\]

Then \([a, a^{\dagger}] = [b, b^{\dagger}] = 1, [a, b] = [a^{\dagger}, b^{\dagger}] = [a^{\dagger}, b] = [a^{\dagger}, b^{\dagger}] = 0\), and

\[
a^{\dagger} a = \frac{1}{2}(a_{x}^{\dagger} a_{x} + a_{y}^{\dagger} a_{y} + ia_{y}^{\dagger} a_{x} - ia_{x}^{\dagger} a_{y}),
\]

\[
b^{\dagger} b = \frac{1}{2}(a_{x}^{\dagger} a_{x} + a_{y}^{\dagger} a_{y} - ia_{y}^{\dagger} a_{x} + ia_{x}^{\dagger} a_{y}),
\]

and for the symmetric case (i.e., for \( \omega_{x} = \omega_{y} = \omega_{o} \)),

\[
\mathcal{H} = (a^{\dagger} a + b^{\dagger} b + 1) \hbar \omega_{o} = (n_{a} + n_{b} + 1) \hbar \omega_{o} = (n + 1) \hbar \omega_{o},
\]

where \( n_{a} \) and \( n_{b} \) are the number of excitations in the rotational states, and \( n = n_{a} + n_{b} = 0, 1, 2, \ldots \). Note now that

\[
L_{z} = (a^{\dagger} a - b^{\dagger} b) \hbar = (n_{a} - n_{b}) \hbar = m \hbar,
\]
so that the eigenvalues of $\mathbf{L}_z$ is $m = n_a - n_b$. This means that for each $n = n_a + n_b$, there are $n + 1$ possible values of $m = n_a - n_b$. It is then easy to see that for a given energy level $E_n$, the quantum number $m$ takes values $m = -n, -n + 2, \ldots, n + 2, n$. 