Drude model

The simplest model for charge carrier transport in metals and semiconductors is the Drude model. In the presence of an applied electric field $\mathbf{E}$, the charge current density current $\mathbf{j}$ is

$$\mathbf{j} = n q \mathbf{v}_d,$$

(1)

where $n$ is the density of the carriers of charge $q$ each, and $\mathbf{v}_d$ is the average carrier drift velocity. The direction of $\mathbf{j}$ is from (+)ve to (-)ve electrode; for electrons, $q = -e$, where $e$ is the magnitude of the electronic charge, and the direction of $\mathbf{v}_d$ is opposite to $\mathbf{j}$ the latter being along $\mathbf{E}$.

If one neglects the thermal motion of the charge carriers, then the equation of motion of an average carrier of mass $m$ and charge $q$ is

$$\frac{d}{dt}(m \mathbf{v}_d) + \frac{m \mathbf{v}_d}{\langle \tau_m \rangle} = q \mathbf{E}.$$  

(2)

The second term above represents a friction, which the carriers experience while drifting through the sample. The friction arises from scattering of the carriers by various sources, such as other carriers, impurities, defects, the vibration of the lattice etc. The main source of friction is usually the vibrating ions/atoms in the lattice, which depends on the lattice temperature, and this is taken care of by a temperature dependent $\langle \tau_m \rangle$, the average momentum relaxation time. The name relaxation means return to equilibrium; and it follows from the fact that, if the electric field is switched off at $t = 0$, the equation of motion for $m \mathbf{v}_d$ has the solution

$$m \mathbf{v}_d(t) = (m \mathbf{v}_d)_{t=0} e^{-t/\langle \tau_m \rangle},$$

(3)

which shows that the drift velocity relaxes exponentially to zero; $\langle \tau_m \rangle$ is usually too short for observing this directly.

In the steady state, the first term in Eqn.(2) vanishes, and the drift velocity is proportional to $|\mathbf{E}|$:

$$|\mathbf{v}_d| \propto |\mathbf{E}| = \mu |\mathbf{E}|,$$

(4)

where the proportionality constant $\mu$, called mobility, is then defined by

$$\mu = \frac{|q|}{m} < \tau_m >.$$  

(5)

This leads to

$$\mathbf{j} = \sigma \mathbf{E} = n q \mu \mathbf{E},$$

(6)

where $\sigma$ is the conductivity defined

$$\sigma = \frac{n q^2}{m} < \tau_m >.$$  

(7)

This formula is known as Drude-conductivity formula. In semiconductors, $m$ is replaced by the effective mass $m^*$ of the carrier. Clearly, the friction term leads to Ohm's Law, i.e., $\sigma$ is independent of the electric field strength.

In elementary derivations of Drude formula found in introductory text books of Solid State Physics, one uses the concept of collision time $\tau_c$. As the carriers drift through the solid, they occasionally collide, for instance, with the vibrating atoms/ions in the lattice, and one would anticipate the existence of a mean free time between the collisions or scattering. The average time interval between successive collisions is the collision time $\tau_c$, which is of the order of $\langle \tau_m \rangle$.

Note that $1/\sigma \equiv \rho$ is the resistivity (with unit $\Omega \text{ cm} = \text{Volt cm/Amp}$) so that the mobility $\mu$ has the unit of cm$^2$/(Volt sec). To get an idea about the order of magnitude for $\mu$, note that $e/m_e = 1.76 \times 10^{15}$ cm$^2$/Volt and taking $< \tau_m > \sim 10^{-13}$ sec (inverse of a typical lattice vibrational frequency), one gets $\mu \sim 176$ cm$^2$/Volt sec. In $n$-type Germanium (Ge), $m^* \sim 0.1 m_e$ and $< \tau_m > \sim 10^{-13}$ sec at room temperature, so that $\mu$ for this material at room temperature is $\mu \sim 1760$ cm$^2$/Volt sec, which agrees in order of magnitude with the observed mobility of 3900 cm$^2$/Volt sec.
Average momentum relaxation time:

In general, the momentum relaxation time $\tau_m$ depends on energy, and one may write

$$\tau_m = \tau_o (\beta \epsilon)^r,$$

where $\tau_o$ is the constant of proportionality having the dimension of time, $\beta = (k_B T / m)^{-1}$, with $k_B$ denoting the Boltzmann constant and $T$ the absolute temperature; $\epsilon$ denotes the energy of the carrier. The power $r$ depends on the nature of scattering (for example, for acoustic deformation potential scattering in semiconductors, $r = -3/2$, while for ionized impurity scattering, $r = 3/2$), and so does $\tau_o$ (for example, for acoustic deformation potential scattering, if the mean-free-path is $l_{ac}$, then $\tau_o \approx l_{ac} / v_{rms}$, where $v_{rms} = \sqrt{d (k_B T / m)}$ the root mean square velocity in $d$-dimension, and $l_{ac}$ itself depends on $T$ as $l_{ac} \propto 1/T$, so that $\tau_o \propto T^{-3/2}$; for ionized impurity scattering $\tau_o \propto T^{3/2}$.

The fact that $\tau_m$ is in general energy dependent, requires a reformulation of the problem. One knows that there is a velocity distribution $f(\mathbf{v})$ for the carriers. The drift $v_{dz}$ for an applied electric field $\mathbf{E}$ along $z$-direction should be

$$v_{dz} = \frac{\int v_z f(\mathbf{v}) d^3v}{\int f(\mathbf{v}) d^3v}.$$  

Similarly, one might write,

$$< \tau_m > = \frac{\int \tau_m(\mathbf{v}) f(\mathbf{v}) d^3v}{\int f(\mathbf{v}) d^3v},$$

which, as will be seen later, however is not the quantity that should enter the expression for the mobility. The equation for the distribution function $f$ is

$$\frac{df(\mathbf{v})}{dt} = \left( \frac{\partial f}{\partial t} \right)_{\text{Collision}},$$

where the right hand side arises from collisions or scattering of the carriers, which drive the system to equilibrium once the external field is withdrawn. This is the so called Boltzmann Transport Equation (BTE) in the semi-classical form.

In general the distribution function depends on the position coordinate $\mathbf{r} \equiv (x,y,z)$, the velocity $\mathbf{v} \equiv (v_x, v_y, v_z)$, and the time $t$, so that using $dx/dt = v_x$, $dy/dt = v_y$, $dz/dt = v_z$, $dv_x/dt = (q/m)E_x$, $dv_y/dt = (q/m)E_y$ and $dv_z/dt = (q/m)E_z$, the term $df/dt$ can be written as

$$\frac{df(\mathbf{r}, \mathbf{v}, t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y + \frac{\partial f}{\partial z} v_z + \frac{\partial f}{\partial v_x} \frac{q}{m} E_x + \frac{\partial f}{\partial v_y} \frac{q}{m} E_y + \frac{\partial f}{\partial v_z} \frac{q}{m} E_z.$$  

The first four terms on the right hand side are zero for dc field, and if there is no temperature or carrier concentration gradient in the sample. In addition, if $E_x = 0 = E_y$, then the Boltzmann equation (11) above takes the form

$$\frac{\partial f}{\partial v_z} \frac{q}{m} E_z = \left( \frac{\partial f}{\partial t} \right)_{\text{Collision}}.$$  

Since $\tau_m$ is a function of velocity, it is natural to write

$$\left( \frac{\partial f}{\partial t} \right)_{\text{Collision}} = -\frac{f(\mathbf{v}) - f_{eq}(\mathbf{v})}{\tau_m(\mathbf{v})},$$

where $f_{eq}(\mathbf{v})$ is the velocity distribution of the carriers in thermal equilibrium in the absence of the electric field $\mathbf{E}$. This then gives the simplest form of BTE in the relaxation time approximation, as

$$\frac{\partial f}{\partial v_z} \frac{q}{m} E_z = -\frac{f(\mathbf{v}) - f_{eq}(\mathbf{v})}{\tau_m(\mathbf{v})},$$

(15)
which leads to

\[ f(v) = f_{eq}(v) - \frac{q}{m} \tau_m(v) \frac{\partial f}{\partial v_z} E_z. \]  

(16)

Note that this is a nonlinear equation for \( f \). However, for small field (i.e., \( E_z \)) one may retain terms linear in \( E_z \) only, i.e., in \( \partial f / \partial v_z \), replace \( f \) by \( f_{eq} \) so that

\[ f(v) \approx f_{eq}(v) + f^{(1)}(v), \quad \text{with} \quad f^{(1)}(v) = -\frac{q}{m} \tau_m(v) \frac{\partial f_{eq}}{\partial v_z} E_z. \]  

(17)

In above, \( f^{(1)}(v) \) denotes the change in the distribution function to linear or first order in the applied electric field.

Using this linearized form described by Eqn.(14) in Eqn.(9), one gets

\[ v_{dz} = \frac{\int v_z f(v) \, d^3v}{\int f(v) \, d^3v} = \frac{\int v_z [f_{eq}(v) + f^{(1)}(v)] \, d^3v}{\int f_{eq}(v) \, d^3v}. \]  

(18)

For a uniform isotropic sample (usually assumed), \( f_{eq}(v) = f_{eq}(v) \), and in that case, the term containing \( v_z f_{eq} \) in the numerator vanishes (because this term is odd function of \( v_z \)) on integration, and one gets to linear order in the electric field,

\[ v_{dz} = \frac{\int v_z f^{(1)}(v) \, d^3v}{\int f_{eq}(v) \, d^3v} = \frac{q}{m} E_z \frac{\int v_z \tau_m(v) \left[-\frac{\partial f_{eq}}{\partial v_z}\right] \, d^3v}{\int f_{eq}(v) \, d^3v}. \]  

(19)