Recall that while introducing the cross-sections, an imaginary sphere of large radius surrounding the particle was drawn, and then considering rates $W$ of energy flow across this sphere, one defined the cross sections as

\[
C_{\text{ext}} = \frac{W_{\text{ext}}}{I_i} = \frac{1}{I_i} \left[ - \int_{\text{sphere}} \vec{S}_{\text{ext}} \cdot \hat{e}_r \, dA \right],
\]

\[
C_{\text{sca}} = \frac{W_s}{I_i} = \frac{1}{I_i} \left[ \int_{\text{sphere}} \vec{S}_s \cdot \hat{e}_r \, dA \right],
\]

where $I_i$ is the incident irradiance and $\vec{S}$s are the time averaged Poynting vectors:

\[
\vec{S}_s = \frac{1}{2} \text{Re} [\vec{E}_s \times \vec{H}_s^*], \quad \text{and} \quad \vec{S}_{\text{ext}} = \frac{1}{2} \text{Re} [\vec{E}_i \times \vec{H}_s^* + \vec{E}_s \times \vec{H}_i^*].
\]

In the general treatment, the sphere radius was assumed to be large so that the far-field asymptotic forms for the scattered field could be used. For spherical particle under consideration, these integrals can be evaluated exactly, and there is no real need to assume the sphere radius to be large.

Thus, using the forms for the fields for the incident $\vec{x}$-polarized light, one has

\[
W_{\text{ext}} = \frac{1}{2} \text{Re} \int_0^{2\pi} d\phi \int_0^\pi d\theta r^2 \sin \theta \left[ E_{i\phi} H_{s\theta}^* - E_{i\theta} H_{s\phi}^* - E_{s\theta} H_{i\phi}^* + E_{s\phi} H_{i\theta}^* \right],
\]

\[
W_s = \frac{1}{2} \text{Re} \int_0^{2\pi} d\phi \int_0^\pi d\theta r^2 \sin \theta \left[ E_{s\theta} H_{s\phi}^* - E_{s\phi} H_{s\theta}^* \right],
\]

where $r \geq R$ is arbitrary.

Now using series expansions and integrating term by term, one gets

\[
W_s = \frac{\pi |E_o|^2}{q \omega \mu} \sum_{l=1}^\infty (2l + 1) \left[ |a_l|^2 + |b_l|^2 \right] \left( \text{Re} [g_l] \right).
\]

where $g_l = -i \xi_l^* \xi_l$ For $r \geq R$, the medium is non-absorbing and $\rho = qr$ is real so that the function $\xi_l$ is also real, which implies that $\text{Re} [g_l] = 1$. Therefore

\[
C_{\text{sca}} = \frac{W_s}{I_i} = \frac{2\pi}{q^2} \sum_{l=1}^\infty (2l + 1) \left[ |a_l|^2 + |b_l|^2 \right].
\]

Proceeding in a similar manner, one finds

\[
C_{\text{ext}} = \frac{W_{\text{ext}}}{I_i} = \frac{2\pi}{q^2} \sum_{l=1}^\infty (2l + 1) \text{Re} [a_l + b_l].
\]

The Fig. 6.6. shows a typical extinction efficiency curve for water droplet of radius $R$. There three main features:

- A series of broad regularly spaced maxima and minima called "interference structures" which oscillate approximately about 2. This originates from the interference between the incident and scattered light.

- Irregular fine structure called "ripple structure". This originates from the near vanishing of the denominators of the coefficients $a_l$ and $b_l$ (corresponding to the excitation of possibly virtual natural modes.)
Now consider the amplitude scattering matrix. Recall the definition

\[
\left( \begin{array}{c}
E_{||s} \\
E_{\perp s}
\end{array} \right) = \frac{e^{iq(r-z)}}{-iqr} \left( \begin{array}{cc}
S_2 & S_3 \\
S_4 & S_1
\end{array} \right) \left( \begin{array}{c}
E_{||i} \\
E_{\perp i}
\end{array} \right),
\]

for \( qr << 1 \) (far field region), where \( S_j \) are to be determined. Also recall that \( \hat{e}_{||s} = \hat{e}_o \), \( \hat{e}_{\perp s} = -\hat{e}_o \), \( E_{||i} = \cos \phi E_{xi} + \sin \phi E_{yi} \), and \( E_{\perp i} = \sin \phi E_{xi} - \cos \phi E_{yi} \). For \( x \)-polarized incident light (considered so far), \( E_{||i} = \cos \phi E_o e^{iqz} \) and \( E_{\perp i} = \sin \phi E_o e^{iqz} \). Therefore (using \( E_{||s} = E_{st} \) and \( E_{\perp s} = -E_{sd} \)), one gets

\[
\left( \begin{array}{c}
E_{s\theta} \\
E_{s\phi}
\end{array} \right) = \frac{e^{iq(r-z)}}{-iqr} E_o e^{iqz} \left( \begin{array}{cc}
S_2 & S_3 \\
S_4 & S_1
\end{array} \right) \left( \begin{array}{c}
\cos \phi \\
\sin \phi
\end{array} \right),
\]

i.e.,

\[
E_{s\theta} = \frac{e^{i\rho}}{-i\rho} E_o \left[ S_2 \cos \phi + S_3 \sin \phi \right],
\]

\[
E_{s\phi} = \frac{e^{i\rho}}{i\rho} E_o \left[ S_4 \cos \phi + S_1 \sin \phi \right],
\]

where \( \rho = qr \). But as shown earlier

\[
E_{s\theta} = -\frac{\cos \phi}{\rho} \sum_{l=1}^{\infty} E_l \left[ b_l \xi_l \pi_l - i a_l \xi'_l \tau_l \right],
\]

\[
E_{s\phi} = -\frac{\sin \phi}{\rho} \sum_{l=1}^{\infty} E_l \left[ i a_l \xi'_l \pi_l - b_l \xi_l \tau_l \right],
\]

and \( \xi_l(\rho) = \rho_l^{(1)}(\rho) \xrightarrow{\rho \gg l^2} -i(-i)^l e^{i\rho} \), so that in the far field region (\( \rho >> l^2 \)),

\[
\xi_l(\rho) \xrightarrow{\rho \gg l^2} \xi_l(\rho) ,
\]
\[
E_{s\theta} = \frac{\cos \phi}{-i \rho} e^{i \rho} \sum_{l=1}^{\infty} E_l \left( -i \right)^l \left[ a_l \tau_l + b_l \pi_l \right] = \cos \phi \frac{e^{i \rho}}{-i \rho} E_0 \sum_{l=1}^{\infty} \frac{(2l + 1)}{l(l + 1)} \left[ a_l \tau_l + b_l \pi_l \right],
\]
\[
E_{s\phi} = \frac{\sin \phi}{i \rho} e^{i \rho} \sum_{l=1}^{\infty} E_l \left( -i \right)^l \left[ a_l \pi_l + b_l \tau_l \right] = \sin \phi \frac{e^{i \rho}}{i \rho} E_0 \sum_{l=1}^{\infty} \frac{(2l + 1)}{l(l + 1)} \left[ a_l \pi_l + b_l \tau_l \right].
\]

Using \( E_l = \left( -i \right)^l \frac{(2l+1)}{l(l+1)} E_c \), it then follows that, \( S_3 = 0 = S_4 \) (for \( x \)-polarized incident light), and
\[
S_1 = \sum_{l=1}^{\infty} \frac{(2l + 1)}{l(l + 1)} \left[ a_l \pi_l + b_l \tau_l \right],
\]
\[
S_2 = \sum_{l=1}^{\infty} \frac{(2l + 1)}{l(l + 1)} \left[ a_l \tau_l + b_l \pi_l \right].
\]

Note that it has been tacitly assumed that the series can be terminated at \( l_{\max} \) and \( qr >> l_{\max}^2 \). Now from the definitions, \( \pi_l(1) = \tau_l(1) = \frac{1}{2} l(l + 1) \) (left as an exercise), in the forward direction,
\[
S(\theta = 0^\circ) \equiv S_1(\theta = 0^\circ) = S_2(\theta = 0^\circ) = \frac{1}{2} \sum_{l=1}^{\infty} (2l + 1) [a_l + b_l].
\]

Using this in the general expression,
\[
C_{ext} = \frac{4\pi}{q^2} \text{Re}[\langle \hat{X} \cdot \hat{\varepsilon}_x \rangle_{\theta=0^\circ}] = \frac{4\pi}{q^2} \text{Re}[S(\theta = 0^\circ)].
\]
on gets
\[
C_{ext} = \frac{2\pi}{q^2} \sum_{l=1}^{\infty} (2l + 1) \text{Re}[a_l + b_l],
\]
the same expression derived earlier.

The relation between incident and scattered stokes parameters are
\[
\begin{pmatrix}
I_s \\
Q_s \\
U_s \\
V_s
\end{pmatrix} = \begin{pmatrix}
I_1 \\
S_{12} \\
S_{21} \\
S_{33}
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
I_i \\
Q_i \\
U_i \\
V_i
\end{pmatrix},
\]

where
\[
S_{11} = \frac{1}{2} [S_2]^2 + |S_1|^2, \quad S_{12} = \frac{1}{2} [S_2]^2 - |S_1|^2, \\
S_{33} = \frac{1}{2} [S_1 S_2^* + S_2 S_1^*], \quad S_{34} = \frac{1}{2} [S_1 S_2^* - S_2 S_1^*],
\]
with \( S_{11}^2 = S_{12}^2 + S_{33}^2 + S_{34}^2 \) so that only three are independent elements.

Consequently, if the incident light is 100% polarized parallel to a particular scattering plane (does not matter which one), then (ignoring the factor \( (q^2 r^2)^{-1} \)), \( I_s = (S_{11} + S_{12}) I_i \), \( Q_s = I_s \), \( U_s = V_s = 0 \), so that the scattered light is also 100% polarized parallel to the scattering plane.

If the incident light is 100% polarized perpendicular to the scattering plane, then (again ignoring the factor \( (q^2 r^2)^{-1} \)),
\[
I_s = (S_{11} - S_{12}) I_i , \quad Q_s = -I_s , \quad U_s = 0 = V_s .
\]

so that the scattered light is also 100% polarized perpendicular to the scattering plane.

Let
\[
i_\parallel = (S_{11} + S_{12}) = |S_2|^2, \quad \text{and} \quad i_\perp = S_{11} - S_{12} = |S_1|^2.
\]
Then for unpolarized incident light, \( I_s = S_{11} I_i ; Q_s = S_{12} I_i ; U_s = V_s = 0 \) and the ratio
\[ P = -\frac{S_{12}}{S_{11}} = \frac{i_\perp - i_\parallel}{i_\perp + i_\parallel}. \]

is such that \(|P| \leq 1\). If \(P\) is positive, the scattered light is partially polarized perpendicular to the scattering plane; if \(P\) is negative then the scattered light is partially polarized parallel to the scattering plane. \(|P|\) is the degree of polarization. Note that irrespective of size and composition of the sphere, \(P(0^\circ) = 0 = P(180^\circ)\). Thus \(P\) gives the details of the nature of polarization of the plane incident wave scattered by a sphere.