Lecture 2: The solution for the fields

These orthogonality properties can now be used to write the em fields.

For example, for a plane $x$-polarized wave:

$$\vec{E}_i = E_o \hat{e}_x e^{i\sigma \cos \theta} , \quad (\hat{e}_x = \sin \theta \cos \phi \hat{e}_r + \cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi) .$$

$$= \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} \left[ B_{eml} \vec{M}_{eml}^* + B_{oml} \vec{M}_{oml}^* + A_{eml} \vec{N}_{eml} + A_{oml} \vec{N}_{oml}^* \right] .$$

Now using the orthogonality properties of $\vec{M}$, $\vec{N}$, etc., one gets

$$B_{eml} = \frac{\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \vec{E}_i \cdot \vec{M}_{eml}^* \right)}{\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta |\vec{M}_{eml}|^2} ,$$

and similar expressions for $B_{oml}$, $A_{eml}$, and $A_{oml}$. Now plugging in the plane wave form in the numerator, one finds on $\phi$ integration that $B_{eml} = 0 = A_{oml}$ for all $m$ and $l$. Further the remaining coefficients vanish unless $m = 1$. Note also that the incident field is finite at the origin which implies $j_l(qr)$ be associated with the vector harmonics, which will signified by the superscript $(1)$. Thus

$$\vec{E}_i = \sum_{l=1}^{\infty} \left[ B_{01l} \vec{M}_{01l}^{(1)} + A_{el1} \vec{N}_{el1}^{(1)} \right] , \quad (Z_l^{(1)} = j_l(\rho)) , \quad \rho = qr .$$

$B_{01l}$ and $A_{el1}$ can be evaluated explicitly (left as problem), and one gets

$$B_{01l} = E_l , \quad A_{el1} = -i E_l , \quad E_l = E_o \frac{(2l + 1)}{l(l + 1)} i^l ,$$

so that

$$\vec{E}_i = \sum_{l=1}^{\infty} E_l \left[ \vec{M}_{01l}^{(1)} - i \vec{N}_{el1}^{(1)} \right] .$$

The corresponding magnetic field is

$$\vec{H}_i = \frac{1}{i \omega \mu} \vec{\nabla} \times \vec{E}_i = -\frac{i q}{\omega \mu} \left[ \frac{1}{q} \vec{\nabla} \times \vec{E}_i \right] = -\frac{i q}{\omega \mu} \sum_{l=1}^{\infty} E_l \left[ \vec{N}_{el1}^{(1)} - i \vec{M}_{el1}^{(1)} \right] .$$

Or

$$\vec{H}_i = -\frac{q}{\omega \mu} \sum_{l=1}^{\infty} E_l \left[ \vec{M}_{el1}^{(1)} + i \vec{N}_{el1}^{(1)} \right] .$$

One can expand the scattered em fields ($\vec{E}_s$, $\vec{H}_s$) and the fields ($\vec{E}_i$, $\vec{H}_i$) inside the sphere (of radius $R$, say) in terms of the vector spherical harmonics. At the boundary between the sphere and the surrounding medium, one has to use the boundary conditions that the tangential components of $\vec{E}$ and $\vec{H}$ must match across the boundary, i.e.,

$$(\vec{E}_i + \vec{E}_s - \vec{E}_1) \times \hat{e}_r = 0 , \quad \text{and} \quad (\vec{H}_i + \vec{H}_s - \vec{H}_1) \times \hat{e}_r = 0 .$$

The boundary conditions, the orthogonality of the vector spherical harmonics, and the form of the expansion of the incident field determine the form of the expansions for the scattered fields and the fields inside the sphere. For instance, for the incident plane wave ($\vec{E}_i$, $\vec{H}_i$), the coefficients of these expansions vanish for all $m \neq 1$. Finiteness of the fields at the origin (center of the sphere) requires that $Z_l$ be taken as $j_l(q_1 r)$ which will be denoted by a superscript (2) (note that here the argument contains $q_1$ instead of $q$ for $Z_l$ labeled earlier with superscript (1)).

Thus one has

$$\vec{E}_1 = \sum_{l=1}^{\infty} E_l \left[ c_l \vec{M}_{01l}^{(2)} - id_l \vec{N}_{el1}^{(2)} \right] .$$
\[ \tilde{H}_1 = -\frac{q_l}{\omega \mu} \sum_{l=1}^{\infty} E_l \left[ d_l \tilde{M}_{e1l}^{(2)} + ic_l \tilde{N}_{o1l}^{(2)} \right] . \]

where the coefficients are yet to be determined. In the region outside the sphere, \( j_l \) and \( y_l \) are both well behaved; however, it is convenient to use the spherical Hankel functions particularly because for \( qr >> l^2 \), they have the asymptotic form

\[ h_l^{(1)}(qr) \propto (-i)^l \frac{iqr}{iqr}, \quad \text{(outgoing spherical wave)} \]
\[ h_l^{(2)}(qr) \propto (-i)^l \frac{-iqr}{iqr}, \quad \text{(incoming spherical wave)} \]

Since the far field scattered wave is a spherical outgoing wave, one must use \( h_l^{(1)}(qr) \) for \( Z_l \) and the corresponding vector fields are denoted by the superscript (3). Thus the expansion for the scattered field is

\[ \tilde{E}_s = \sum_{l=1}^{\infty} E_l \left[ ia_l \tilde{N}_{e1l}^{(3)} - b_l \tilde{M}_{o1l}^{(3)} \right] . \]
\[ \tilde{H}_s = \frac{q}{\omega \mu} \sum_{l=1}^{\infty} E_l \left[ ib_l \tilde{N}_{o1l}^{(3)} + a_l \tilde{M}_{e1l}^{(3)} \right] . \]

where the unknown coefficients \( a_l, b_l, c_l \) and \( d_l \) are determined by the boundary conditions. For compactness, define

\[ \pi_l = \frac{1}{\sin \theta} P_l^{m=1}(\cos \theta), \quad \text{and} \quad \tau_l = \frac{d}{d\theta} P_l^{m=1}(\cos \theta), \]

and

\[ \psi_l(\rho) = \rho j_l(\rho), \quad \text{and} \quad \xi_l(\rho) = \rho h_l^{(1)}(\rho). \]

with \( \zeta_l(\rho) = \rho Z_l(\rho) \). Here, \( \pi_l \) and \( \tau_l \) are the \( \theta \) angle dependent functions while \( \psi_l \) and \( \xi_l \) are \( \rho \) dependent functions (called "Riccati-Bessel functions") appearing in \( \tilde{M}_{o1l}, \tilde{M}_{e1l}, \tilde{N}_{o1l} \) and \( \tilde{N}_{e1l} \). Typical shapes of \( \pi_l \) and \( \tau_l \) as function of \( \theta \) are shown in Fig. 6.4.

\[ \text{Fig. 6.4. Polar Plots} \]

Now one can write...
\[ M_{\phi l} = \frac{\zeta_l}{\rho} \tau_l \cos \phi \hat{e}_\theta - \frac{\zeta_l}{\rho} \tau_l \sin \phi \hat{e}_\phi. \]
\[ M_{\theta l} = -\frac{\zeta_l}{\rho} \tau_l \sin \phi \hat{e}_\theta - \frac{\zeta_l}{\rho} \tau_l \cos \phi \hat{e}_\phi. \]
\[ N_{\phi l} = \frac{\zeta_l}{\rho^2} l(l+1) \sin \theta \tau_l \sin \phi \hat{e}_r + \frac{\zeta_l'}{\rho} \tau_l \sin \phi \hat{e}_\theta + \frac{\zeta_l'}{\rho} \tau_l \cos \phi \hat{e}_\phi. \]
\[ N_{\theta l} = \frac{\zeta_l}{\rho^2} l(l+1) \sin \theta \tau_l \cos \phi \hat{e}_r + \frac{\zeta_l'}{\rho} \tau_l \cos \phi \hat{e}_\theta - \frac{\zeta_l'}{\rho} \tau_l \sin \phi \hat{e}_\phi. \]

where prime means derivative with respect to argument, e.g., $\zeta' = d\zeta(\rho)/d\rho$.

Then for the x-polarized incident wave, one obtains (left as exercise)

\[ E_{i\theta} = \frac{\cos \phi}{\rho} \sum_{l=1}^{\infty} E_l (\psi_l \tau_l - i\psi'_l \tau_l), \quad H_{i\theta} = \frac{q}{\omega \mu} \tan \phi E_{i\theta}, \quad \rho = qr. \]
\[ E_{i\phi} = \frac{\sin \phi}{\rho} \sum_{l=1}^{\infty} E_l (i\psi'_l \tau_l - \psi_l \tau_l), \quad H_{i\phi} = -\frac{q}{\omega \mu} \cot \phi E_{i\phi}, \quad \rho = qr. \]
\[ E_{1\theta} = \frac{\cos \phi}{\rho} \sum_{l=1}^{\infty} E_l (c_l \psi_l \tau_l - id_l \psi'_l \tau_l), \quad \rho = q_1 r. \]
\[ E_{1\phi} = \frac{\sin \phi}{\rho} \sum_{l=1}^{\infty} E_l (id_l \psi'_l \tau_l - c_l \psi_l \tau_l), \quad \rho = q_1 r. \]
\[ H_{1\theta} = \frac{q_1}{\omega \mu_1} \frac{\sin \phi}{\rho} \sum_{l=1}^{\infty} E_l (d_l \psi_l \tau_l - ic_l \psi'_l \tau_l), \quad \rho = q_1 r. \]
\[ H_{1\phi} = -\frac{q_1}{\omega \mu_1} \frac{\cos \phi}{\rho} \sum_{l=1}^{\infty} E_l (ic_l \psi'_l \tau_l - d_l \psi_l \tau_l), \quad \rho = q_1 r. \]
\[ E_{s\theta} = -\frac{\cos \phi}{\rho} \sum_{l=1}^{\infty} E_l (b_l \xi_l \tau_l - i a_l \xi'_l \tau_l), \quad \rho = q_1 r. \]
\[ E_{s\phi} = -\frac{\sin \phi}{\rho} \sum_{l=1}^{\infty} E_l (ia_l \xi'_l \tau_l - b_l \xi_l \tau_l), \quad \rho = qr. \]
\[ H_{s\theta} = -\frac{q}{\omega \mu} \frac{\sin \phi}{\rho} \sum_{l=1}^{\infty} E_l (a_l \xi_l \tau_l - ib_l \xi'_l \tau_l), \quad \rho = qr. \]
\[ H_{s\phi} = \frac{q}{\omega \mu} \frac{\cos \phi}{\rho} \sum_{l=1}^{\infty} E_l (ib_l \xi'_l \tau_l - a_l \xi_l \tau_l), \quad \rho = qr. \]

Now one can determine the coefficients $a_l$, $b_l$, $c_l$ and $d_l$ using the boundary conditions (at the sphere radius $r = R$), which in the component form are

\[ E_{i\theta} + E_{s\theta} = E_{1\theta}, \quad E_{i\phi} + E_{s\phi} = E_{1\phi}, \quad r = R. \]
\[ H_{i\theta} + H_{s\theta} = H_{1\theta}, \quad H_{i\phi} + H_{s\phi} = H_{1\phi}, \quad atr = R. \]

The equation $E_{i\theta} + E_{s\theta} = E_{1\theta}$ at $r = R$ gives

\[ \sum_{l=1}^{\infty} E_l [[m(\psi_l(x) - b_l \xi_l(x)) - c_l \psi_l(mx)] \tau_l \]
\[ + [m(\psi'_l(x) - a_l \xi'_l(x)) - d_l \psi'_l(mx)] (-i \tau_l)] = 0. \]

where $m = q_1/q = N_1/N$ the refractive index of the particle relative to that of the surrounding medium, and $x = qR = (2\pi N/\lambda)R$ is the so called "size parameter". This has to be satisfied for
arbitrary values of $\theta$ or $\pi l$ and $\tau l'$ and therefore one must have
\[ m(\psi_1(x) - b_1\xi_1(x)) - c_1\psi_1(mx) = 0, \]
\[ m(\psi'_1(x) - a_1\xi'_1(x)) - d_1\psi'_1(mx) = 0. \]

Or,
\[ ma_1\xi'_1(x) + d_1\psi'_1(mx) = m\psi'_1(x), \]
\[ mb_1\psi_1(x) + c_1\psi_1(mx) = m\psi_1(x), \]

The equation $E_{i\phi} + E_{s\phi} = E_{1\phi}$ at $r = R$ gives the same results as above. Similarly, the equation $H_{i\theta} + H_{s\theta} = H_{1\theta}$ at $r = R$ gives
\[ \frac{\mu_1}{\mu} a_1\xi_1(x) + d_1\psi_1(mx) = \frac{\mu_1}{\mu} \psi_1(x), \]
\[ \frac{\mu_1}{\mu} b_1\psi'_1(x) + c_1\psi'_1(mx) = \frac{\mu_1}{\mu} \psi'_1(x), \]

which are reproduced if one uses the equation $H_{i\phi} + H_{s\phi} = H_{1\phi}$ at $r = R$.

Solving for the coefficients, one gets
\[ a_l = \left( \frac{\psi_1(x)}{\xi_1(x)} \right) \left[ \frac{mL_1^{(\psi)}(x) - \frac{\mu_1}{\mu} L_1^{(\psi)}(mx)}{mL_1^{(\xi)}(x) - \frac{\mu_1}{\mu} L_1^{(\xi)}(mx)} \right], \]
\[ b_l = \left( \frac{\psi_1(x)}{\xi_1(x)} \right) \left[ \frac{mL_1^{(\psi)}(mx) - \frac{\mu_1}{\mu} L_1^{(\psi)}(x)}{mL_1^{(\xi)}(mx) - \frac{\mu_1}{\mu} L_1^{(\xi)}(x)} \right], \]
\[ c_l = \left( \frac{m\mu_1}{\mu} \right) \left( \frac{\psi_1(x)}{\psi_1(mx)} \right) \left[ \frac{L_1^{(\psi)}(x) - L_1^{(\xi)}(x)}{mL_1^{(\psi)}(mx) - \frac{\mu_1}{\mu} L_1^{(\xi)}(mx)} \right], \]
\[ d_l = \left( \frac{m\mu_1}{\mu} \right) \left( \frac{\psi_1(x)}{\psi_1(mx)} \right) \left[ \frac{L_1^{(\psi)}(x) - L_1^{(\xi)}(x)}{mL_1^{(\psi)}(mx) - \frac{\mu_1}{\mu} L_1^{(\xi)}(mx)} \right]. \]

where $L_1^{(\psi)} = \psi'_1(x)/\psi_1(x)$, $L_1^{(\xi)}(mx) = \xi'_1(mx)/\xi_1(mx)$, $L_1^{(\psi)} = \psi'_1(x)/\psi_1(x)$, and $L_1^{(\xi)} = \xi'_1(mx)/\xi_1(mx)$.

Note that the denominators of $a_l$ and $d_l$ are identical while those of $b_l$ and $c_l$ are identical. If for a particular value of $x$ one of these denominators becomes vanishingly small, the corresponding "normal mode" will dominate the scattered field. The $a_l$ mode is dominant if the condition
\[ a_l \text{ modes: } mL_1^{(\xi)}(x) = \frac{\mu_1}{\mu} L_1^{(\psi)}(mx), \]
is approximately satisfied; similarly $b_l$ mode is dominant if the condition
\[ b_l \text{ modes: } mL_1^{(\psi)}(mx) = \frac{\mu_1}{\mu} L_1^{(\xi)}(x), \]
is approximately satisfied. In general the scattered field is a superposition of normal modes. The frequencies for which $a_l$ or $b_l$ denominators become very small are called "natural frequencies" of the sphere; in general these are complex and the associated modes are said to be "virtual". If the imaginary parts of the complex frequencies are small compared with the real parts, then the incident wave excites these modes.
The electromagnetic field are sometimes classified according to the nature of the normal modes. For each \( I_l \), there are two distinct types of modes: one for which there is no radial magnetic component in the scattered field called "Transverse Magnetic (TM) modes", and another for which there is no radial electric field component is called "Transverse Electric (TE) modes"; in the terminology of wave guides, these are called TM and TE modes respectively. For TM modes, \( a_{lL} \) and thereby \( \vec{M}_{e1l} \) dominate in the scattered field, while in TE modes, \( b_{lL} \) and thereby \( \vec{M}_{o1l} \) dominate; since \( \vec{M} \) has no radial component, this makes the magnetic field for the TM modes transverse (but not the electric field which has \( \vec{N} \) with a radial component) and the electric field for the TE mode transverse (here, not the magnetic field which has \( \vec{N} \) component). Typical fields inside the sphere are depicted in Fig. 6.15.