In this module we consider in detail the case of single particle.

The basic equations for em-fields (\(\vec{E}, \vec{H}\)) in a linear and isotropic homogeneous medium are (for time harmonic fields with time dependence \(e^{-i\omega t}\))

\[
\nabla^2 \vec{E} + q^2 \vec{E} = 0, \quad \nabla^2 \vec{H} + q^2 \vec{H} = 0, \quad q^2 = \omega^2 \varepsilon \mu.
\]

\(\nabla \cdot \vec{E} = 0, \quad \nabla \cdot \vec{H} = 0\).

In addition \(\vec{E}\) and \(\vec{H}\) are related by

\[
\nabla \times \vec{E} = i \omega \mu \vec{H}, \quad \nabla \times \vec{H} = -i \omega \varepsilon \vec{E}.
\]

Now suppose that \(\psi\) is a scalar function and \(\vec{C}\) is an arbitrary constant vector. Then construct a vector field \(\vec{M}\) such that

\[
\vec{M} = \nabla \times (\vec{C} \psi) = -\vec{C} \times \nabla \psi.
\]

It is important to note that this holds even for \(\vec{C} = \vec{r}\), the position vector.

By construction, \(\nabla \cdot \vec{M} = 0\) (since divergence of curl vanishes), and \(\vec{M}\) is perpendicular to \(\vec{C}\). Now using the vector identities

\[
\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B},
\]

\[
\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{B} \cdot \nabla) \vec{A} + (\vec{A} \cdot \nabla) \vec{B},
\]

one gets

\[
\nabla^2 \vec{M} + q^2 \vec{M} = \nabla \times \left[ \vec{C} (\nabla^2 \psi + q^2 \psi) \right].
\]

Therefore the vector field \(\vec{M}\) satisfy the vector wave equation

\[
\nabla^2 \vec{M} + q^2 \vec{M} = 0,
\]

if \(\psi\) satisfy the scalar wave equation

\[
\nabla^2 \psi + q^2 \psi = 0.
\]

Next construct another vector field

\[
\vec{N} = \frac{1}{q} \left( \nabla \times \vec{M} \right),
\]

so that \(\nabla \cdot \vec{N} = 0\). Then one finds that \(\vec{N}\) also satisfy the vector wave equation

\[
\nabla^2 \vec{N} + q^2 \vec{N} = 0.
\]

Further, note that \(\nabla \times \vec{N} = q\vec{M}\), so that \(\vec{M}\) and \(\vec{N}\) have all the required properties of the em fields: they satisfy the vector wave equation, their divergence vanish, the curl of \(\vec{M}\) is proportional to \(\vec{N}\) and curl of \(\vec{N}\) proportional to \(\vec{M}\). Thus the problem is reduced to first finding the scalar function \(\psi\) (called "generating function" for the so called "vector harmonics" \(\vec{M}\) and \(\vec{N}\)); the vector \(\vec{C}\) is sometimes called the "guiding" or "pilot" vector.

For the problem of scattering by a sphere, it is convenient to use its symmetry and therefore consider the problem of finding the scalar function in spherical polar coordinates \((r, \theta, \phi)\), and choose the pilot vector \(\vec{C}\) as \(\vec{r}\) (the radius vector), so that \(\vec{M}\) is always tangential to the sphere \(|\vec{r}| = \text{constant}\) (i.e., \(\vec{r} \cdot \vec{M} = 0\)) like the electric field for a scattered em wave.
The scalar wave equation in spherical polar coordinates (see Fig. 6.1) is

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} + q^2 \psi = 0.
\]

The problem is separable, and one can write

\[
\psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi),
\]

with equations

\[
\frac{d^2 \Phi}{d\phi^2} + m^2 \Phi = 0,
\]

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \left[ \frac{i(l+1)}{\sin^2 \theta} - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0.
\]

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \left[ q^2 - \frac{l(l+1)}{r^2} \right] R = 0.
\]

The linearly independent solutions for \( \Phi \) are

\[
\Phi_e = \cos m \phi, \quad \Phi_o = \sin m \phi, \quad (e \equiv \text{even}, \quad o \equiv \text{odd})
\]

where \( m = 0, 1, 2, 3, \ldots \) (\( \Phi_{-m} \) is not linearly independent of \( \Phi_m \)), so that \( \psi \) is single valued, i.e., \( \psi(\phi + 2\pi) = \psi(\phi) \).

The solution for \( \Theta \) which remain finite for \( \theta = 0 \) and \( \theta = \pi \) are \( P_l^m(\cos \theta) \), the well known Associated Legendre functions of the first kind of degree \( l \) and order \( m \), where \( l = m, m + 1, \ldots \).

For \( R \), it is convenient to introduce a new dimensionless variable \( \rho = qr \) and define a function \( Z = R \sqrt{\rho} \). Then the equation for \( R \) is converted to an equation for \( Z \) as

\[
\rho \frac{d}{d\rho} \left( \rho \frac{dZ}{d\rho} \right) + \left[ \rho^2 - \left( l + \frac{1}{2} \right)^2 \right] Z = 0.
\]

The linearly independent solutions for \( Z \) are the Bessel functions of the first kind \( J_{\nu} \) and second kind \( Y_{\nu} \) (sometimes denoted by \( N_{\nu} \)), where \( \nu = l + 1/2 \), the order is half integral. Therefore the linearly independent solutions for \( R \) are the spherical Bessel functions,
\[ j_l(\rho) = \sqrt{\frac{\pi}{2\rho}} J_{l+\frac{1}{2}} \quad \text{and} \quad y_l(\rho) = \sqrt{\frac{\pi}{2\rho}} Y_{l+\frac{1}{2}}; \]

where the constant factor \( \sqrt{\pi/2} \) is introduced to comply with standard definitions. Two linearly independent solutions which can be constructed by taking suitable linear combinations of \( j_l \) and \( y_l \) are the spherical Bessel functions of third kind (also known as spherical Hankel functions):

\[ h^{(1)}_l(\rho) = j_l(\rho) + i y_l(\rho), \quad \text{and} \quad h^{(2)}_l(\rho) = j_l(\rho) - i y_l(\rho), \]

which are relevant for scattering problem, as will be seen later.

Typical plots of \( j_l \) and \( y_l \) are shown in Fig. 6.2 and Fig. 6.3 respectively.

![Fig. 6.2: Typical shape of \( j_l \)](image)

![Fig. 6.3: Typical shape of \( y_l \)](image)

The scalar wave functions \( \psi \) can then be written as

\[ \psi_{eml} = Z_l(qr) P^m_l(\cos \theta) \cos m\phi, \quad \text{and} \quad \psi_{oml} = Z_l(qr) P^m_l(\cos \theta) \sin m\phi, \]

where \( Z_l \) is any of the four spherical Bessel functions \( j_l, y_l, h^{(1)}_l \) and \( h^{(2)}_l \). Because of the completeness of the functions \( \cos m\phi, \sin m\phi, P^m_l(\cos \theta) \) and \( Z_l(qr) \), any function satisfying the scalar wave equation in spherical polar coordinates can be expanded in terms of \( \psi_{eml} \) and \( \psi_{oml} \). The vector spherical
harmonics are then

\[ \vec{M}_{eml} = \vec{\nabla} \times (\vec{r} \psi_{eml}), \quad \vec{M}_{oml} = \vec{\nabla} \times (\vec{r} \psi_{oml}), \]

\[ \vec{N}_{eml} = \frac{1}{q} \vec{\nabla} \times \vec{M}_{eml}, \quad \vec{N}_{oml} = \frac{1}{q} \vec{\nabla} \times \vec{M}_{oml}. \]

which in component form, are

\[ \vec{M}_{eml} = Z_l P_l^m \left( -\frac{m}{\sin \theta} \right) \sin m\phi \hat{e}_\theta + Z_l \left( -\frac{d P_l^m}{d\theta} \right) \cos m\phi \hat{e}_\phi, \]

\[ \vec{M}_{oml} = Z_l P_l^m \left( \frac{m}{\sin \theta} \right) \cos m\phi \hat{e}_\theta + Z_l \left( -\frac{d P_l^m}{d\theta} \right) \sin m\phi \hat{e}_\phi, \]

\[ \vec{N}_{eml} = \frac{Z_l}{\rho} l(l+1) P_l^m \cos m\phi \hat{e}_r + \frac{1}{\rho} \frac{d}{d\rho} \left( \rho Z_l \right) \left( \frac{d P_l^m}{d\theta} \right) \cos m\phi \hat{e}_\ell \]

\[ + \frac{1}{\rho} \frac{d}{d\rho} \left( \rho Z_l \right) \left( -\frac{m}{\sin \theta} \right) P_l^m \sin m\phi \hat{e}_\phi, \]

\[ \vec{N}_{oml} = \frac{Z_l}{\rho} l(l+1) P_l^m \sin m\phi \hat{e}_r + \frac{1}{\rho} \frac{d}{d\rho} \left( \rho Z_l \right) \left( \frac{d P_l^m}{d\theta} \right) \sin m\phi \hat{e}_\ell \]

\[ + \frac{1}{\rho} \frac{d}{d\rho} \left( \rho Z_l \right) \left( \frac{m}{\sin \theta} \right) P_l^m \cos m\phi \hat{e}_\phi. \]

Any solution for the vector field wave equation can be expanded in terms of these. The key to such expansions is the following orthogonality relations:

\[ \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \vec{M}_{eml'} \cdot \vec{M}_{oml} \right) = 0, \]

\[ \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \vec{N}_{eml'} \cdot \vec{N}_{oml} \right) = 0. \]

\[ \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \vec{M}_{oml'} \cdot \vec{N}_{eml} \right) = 0. \]

\[ \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \vec{M}_{eml} \cdot \vec{M}_{eml} \right) = 0. \]

for all \( m, m', l \) and \( l' \), and

\[ \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \vec{M}_{eml'} \cdot \vec{M}_{eml} \right) = 0, \]

\[ \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \vec{M}_{oml'} \cdot \vec{M}_{oml} \right) = 0. \]

\[ \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \vec{N}_{eml'} \cdot \vec{N}_{eml} \right) = 0. \]

\[ \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \vec{N}_{oml'} \cdot \vec{N}_{oml} \right) = 0. \]

for all \( l \neq l' \) and all \( m, m' \).