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Lec - 13
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$\rho(x)$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}. \quad ; \quad \vec{E} = -\nabla\phi$$

$$\boxed{\nabla^2\phi = -\frac{\rho}{\epsilon_0}}$$ Poisson Equation

$$\boxed{\nabla^2\phi = 0}$$ Laplace Equation.

Harmonic Functions

$$\vec{E} = \frac{\Phi_0 ab}{b-a} \cdot \frac{1}{r^2} \hat{r} .$$

$$\sigma_{in} = \epsilon_0 \cdot \frac{\Phi_0 ab}{b-a} \cdot \frac{1}{a^2} .$$

$$Q_{in} = 4\pi\epsilon_0 . \Phi_0 \frac{ab}{b-a} = - Q_{out} .$$

$$C = \frac{4\pi\epsilon_0 ab}{b-a} .$$

If $b >>$ $\boxed{C = 4\pi\epsilon_0 a}$ Single Conductor.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = 0$$

$$r^2\frac{\partial\Phi}{\partial r} = A$$

$$\Phi = -\frac{A}{r} + B$$

$$\Phi(r=b) = 0 = -\frac{A}{b} + B \qquad \boxed{B = \frac{A}{b}}$$

$$\Phi(r=a) = \Phi_0 = -\frac{A}{a} + \frac{A}{b} = A\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$A = -\frac{\Phi_0 ab}{b-a}$$

$$\Phi(r) = \frac{\Phi_0 ab}{b-a}\cdot\frac{1}{r} - \frac{\Phi_0 a\cancel{b}}{b-a}\cdot$$

$$\Phi(r) = \frac{\Phi_0 ab}{b-a}\left(\frac{1}{r} - \frac{1}{b}\right)$$

$$\vec{E} = -\nabla\phi = -\frac{\partial}{\partial\rho}\varphi \cdot \hat{\rho}$$

$$= \frac{\varphi_0}{\ln(a/b)} \cdot \frac{1}{\rho}\,\hat{\rho}\,.$$

$$\sigma_{\text{inside}} = \epsilon_0 E_n = \frac{\epsilon_0 \varphi_0}{a \ln(a/b)}$$

$$\underline{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right)} = 0$$

$$\rho\frac{\partial\varphi}{\partial\rho} = A$$

$$\frac{\partial\varphi}{\partial\rho} = \frac{A}{\rho} \Rightarrow \boxed{\varphi = A\ln\rho + B} \leftarrow$$

$$\varphi(b) = 0 \qquad 0 = A\ln \cancel{\rho}^{\,b} + B$$

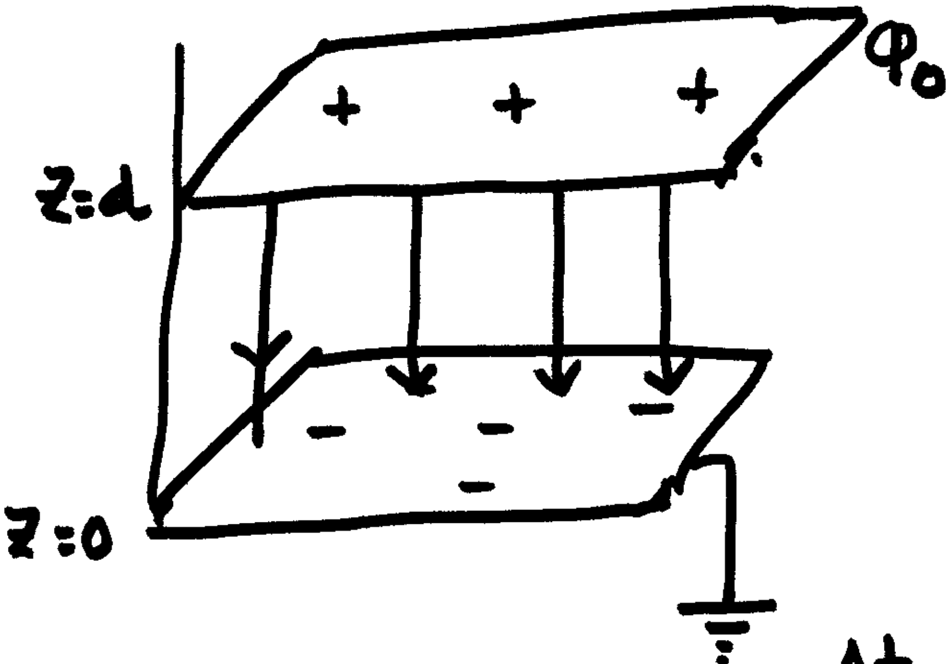
$$B = -A\ln \cancel{\rho}\, b$$

$$\varphi(a) = \varphi_0 \qquad \varphi_0 = A\ln a - A\ln b.$$

$$A = \frac{\varphi_0}{\ln(a/b)}.$$

$$\varphi = \frac{\varphi_0\ln(\rho/b)}{\ln(a/b)}$$

Parallel plate Capacitor



$$\nabla^2\varphi = \frac{d^2\varphi}{dz^2} = 0$$

$$\frac{d\varphi}{dz} = A$$

$$\varphi(z) = Az + B.$$

At $z=0$ $\quad \varphi = 0$, $B=0$

$z=d$ $\quad \varphi = \varphi_0 \Rightarrow A = \frac{\varphi_0}{d}$

$$\varphi(z) = \frac{\varphi_0}{d}.z \quad ., \quad \vec{E} = -\frac{\varphi_0}{d}\hat{k}.$$

$$Q = \frac{A\varphi_0}{d}\epsilon_0.$$

$$= C\varphi_0$$

$$C = \frac{A\epsilon_0}{d}.$$

$\sigma = \epsilon_0 E_n$ $\quad$ Lower plate : $\sigma = -\frac{\varphi_0}{d}.\epsilon_0.$

Upper plate : $\sigma = +\frac{\varphi_0}{d}.\epsilon_0.$

Green's First Identity

φ, ψ arbitrary scalar fns

$$\int \left(\varphi \nabla^2 \psi + \overset{\nabla\psi \cdot \nabla\phi}{\cancel{\psi \nabla \varphi}} \right) d^3r = \oint_S \varphi \frac{\partial \psi}{\partial n} \, ds .$$

$$\varphi = \psi = \Phi \longrightarrow \Phi_1 - \Phi_2 .$$

$$\int \left(\Phi \nabla^2 \Phi + \nabla\Phi \cdot \nabla\Phi \right) d^3r = \underline{\underline{\oint \Phi \frac{\partial \Phi}{\partial n}}} = 0$$

$$\int \left(\Phi \nabla^2 \phi + |\nabla \phi|^2 \right) d^3r = 0$$

$$\int |\nabla \phi|^2 d^3r = 0$$

$$\nabla \phi = 0 \Rightarrow \Phi_1 = \Phi_2$$

$\underline{\underline{\varphi_1}}$, $\underline{\underline{\varphi_2}}$

$$\varphi = \varphi_1 - \varphi_2 .$$

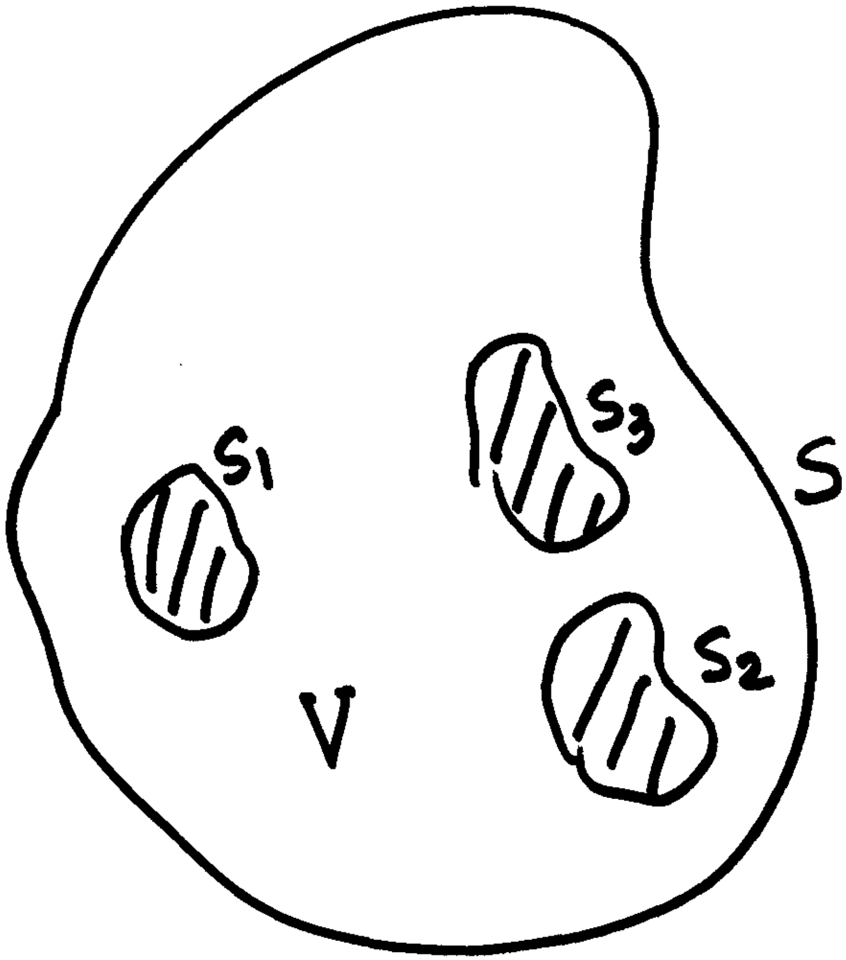
$$\nabla^2 \varphi_1 = 0$$

$$\nabla^2 \dot{\varphi}_2 = 0$$

$$\nabla^2 \varphi = \nabla^2 (\varphi_1 - \varphi_2) = 0$$

$$\varphi_1(S_1) = \varphi_2(S_1) \quad \Bigg| \quad \left.\frac{\partial \varphi_1}{\partial n}\right|_{S_1} = \left.\frac{\partial \varphi_2}{\partial n}\right|_{S_2}$$

$$\varphi(S_1, S_2 \cdots) = 0 \quad \Bigg| \quad \left.\frac{\partial \varphi}{\partial n}\right|_{S_1 \ldots} = 0$$



Boundary Condition

Specify φ at $S_1, S_2 \ldots$ $\rightarrow$ $\boxed{\text{Dirichlet}}$

$\dfrac{\partial \varphi}{\partial n}$ $\rightarrow$ $\boxed{\text{Neumann}}$

Cylindrical (ρ, θ, z)

$$\nabla^2 \varphi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \varphi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} \quad \|$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{Space}} \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\underline{\underline{r}}'$$

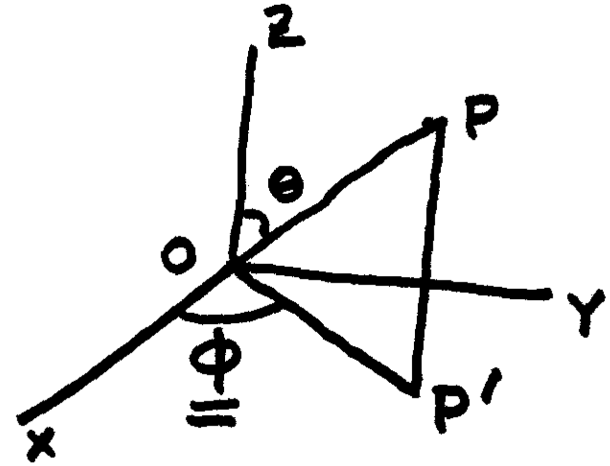
$$\nabla^2 \varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}\,')\, \underbrace{\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|}}\, d^3 r'$$

$$= \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}\,')\left(-4\pi\,\delta^3(\underline{\underline{\vec{r}-\vec{r}}}\,')\right) d^3 r'$$

$$= -\frac{1}{\epsilon_0}\rho(\vec{r})$$

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

<u>Spherical</u> (r, θ, ϕ)



$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)$$
$$+ \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)$$
$$+ \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\varphi}{\partial\phi^2}$$