Module II: Relativity and Electrodynamics
Lecture 8: EM field tensor and Maxwell’s equations

Amol Dighe
TIFR, Mumbai
The electromagnetic field tensor $F$

Maxwell’s equations in terms of $F$ and $\tilde{F}$
The electromagnetic field tensor $F$

Maxwell’s equations in terms of $F$ and $\tilde{F}$
Motivation for the EM field tensor $F$

- We have seen that the electromagnetic potential $A$ is a 4-vector, and hence a useful quantity to deal with relativistically. However, it is a gauge-dependent quantity, and hence does not have a unique value. The following changes in $A$ does not change the physics:

\[
\begin{align*}
\phi & \to \phi + \partial \psi / \partial t \\
\vec{A} & \to \vec{A} - \nabla \psi
\end{align*}
\]

\[\Rightarrow \quad A^m \to A^m + \partial^m \psi \quad (1)\]

where $\psi$ is any scalar function.

- Note that $\vec{E}$ and $\vec{B}$, the physically measurable quantities, are written in terms of derivatives of $\phi$ and $\vec{A}$. We therefore expect that there will be some quantity that is expressed in terms of derivatives of $A$ that will not change under the above gauge transformation.

- The EM field tensor is such a quantity (indeed, the simplest such nontrivial quantity that includes all the information in $A$):

\[F^{mn} = \partial^m A^n - \partial^n A^m \quad (2)\]
F in terms of $\vec{E}$ and $\vec{B}$

- Note that
  \[
  F^0\alpha = \partial^0 A^\alpha - \partial^\alpha A^0
  \]
  i.e.
  \[
  \vec{V} = \frac{\partial}{\partial (ct)} \vec{A} + \nabla \frac{\phi}{c} = -\frac{\vec{E}}{c},
  \]
  where $\vec{V}$ is the vector part of the rank-2 antisymmetric tensor $F$ as discussed earlier. (Note: we shall use $c = 1$ from henceforth in this lecture.)

- For the purely space components of $F$, one gets
  \[
  F^{12} = -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = -B_z, \quad \text{etc.}
  \]

- The net expression for the covariant components of $F$ is
  \[
  F^{km} = \begin{pmatrix}
  0 & -E_x & -E_y & -E_z \\
  E_x & 0 & -B_z & B_y \\
  E_y & B_z & 0 & -B_x \\
  E_z & -B_y & B_x & 0 
  \end{pmatrix}.
  \]

\(3\)
Different representations of $F$

- Since $F$ has a central role in electrodynamics, it is a good idea to be familiar with its different representations. For example, the covariant and mixed components of $F$ are:

$$F_{km} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \quad F^m_k = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

- The dual pseudotensor $\tilde{F}$ that has the same information as $F$ can be written as

$$\tilde{F}_{km} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix}, \quad \tilde{F}^k_m = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$
Lorentz invariants with $F$ and $\tilde{F}$

- With only $F$, one can form the Lorentz invariant

$$F^{km}F_{km} = 2(|\vec{B}|^2 - |\vec{E}|^2) \quad (4)$$

This is a scalar quantity, and indicates that there are limits on how much the relative values of $\vec{B}$ and $\vec{E}$ can change. In particular,

- If $|\vec{B}| = |\vec{E}|$ in one frame, $|\vec{B}| = |\vec{E}|$ in all frames.
- If $|\vec{B}| > |\vec{E}|$ in one frame, $|\vec{B}| > |\vec{E}|$ in all frames, and vice versa.

- With $F$ and $\tilde{F}$, one can also form

$$F^{km}\tilde{F}_{km} = 4\vec{E} \cdot \vec{B} \quad (5)$$

which is a pseudoscalar. This also puts severe constraints on $\vec{E}$ and $\vec{B}$ in different frames. In particular,

- If $\vec{E} \perp \vec{B}$ in one frame, $\vec{E} \perp \vec{B}$ in all frames.
- If the angle between $\vec{E}$ and $\vec{B}$ is acute ($\vec{E} \cdot \vec{B} > 0$) in one frame, it stays acute in all frames. Similarly for obtuse angles ($\vec{E} \cdot \vec{B} < 0$).
Coming up...

The electromagnetic field tensor $F$

Maxwell’s equations in terms of $F$ and $\tilde{F}$
The source-free equation: $\partial_l \tilde{F}^{lm} = 0$

- Since $F_{ik} = \partial_i A_k - \partial_k A_i$ is an antisymmetric tensor, it trivially satisfies the identity
  \[
  \partial_\ell F_{ik} + \partial_i F_{k\ell} + \partial_k F_{\ell i} = 0 .
  \]

- Multiplying this equation by $\epsilon^{iklm}$ gives
  \[
  \partial_l \tilde{F}^{lm} = 0 . \tag{6}
  \]

- This corresponds to two of the Maxwell’s equations:
  \[
  \partial_\alpha \tilde{F}^{\alpha 0} = 0 \Rightarrow \nabla \cdot \vec{B} = 0 ,
  \]
  \[
  \partial_\alpha \tilde{F}^{\alpha \beta} = 0 \Rightarrow \frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 . \tag{7}
  \]

- Thus, the source-free equations of Maxwell emerge simply from the definition of $F$. This is not surprising, since even in the non-relativistic analysis before, these equations are trivially satisfied once $\vec{A}$ and $\phi$ exist that lead to $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla \phi - \partial \vec{A}/\partial t$. 


Maxwell’s equations with sources

- The Maxwell’s equation we just obtained contained information about the derivative of $\tilde{F}$. It would be interesting to check what the derivatives of $F$ itself are. Therefore we calculate $\partial_m F^{mn}$.

- We have

\[
\begin{align*}
\partial_m F^{m0} &= \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \\
\partial_m F^{m\beta} &= -\frac{\partial E_\alpha}{\partial t} + (\nabla \times \vec{B})_\alpha = \mu_0 J_\alpha
\end{align*}
\]

These two equations may be combined into (using $\epsilon_0 \mu_0 = 1/c^2 = 1$)

\[
\partial_m F^{mn} = \mu_0 J^n .
\]

- The above represent the remaining two Maxwell’s equations in terms of the EM field tensor $F$ and the sources $J$. 

The electromagnetic field tensor $F$ is the logical choice for an object with the right transformation properties whose elements are uniquely measurable (components of $\vec{E}$ and $\vec{B}$).

Maxwell’s four equations can be written in a compact form in terms of $F$ as two equations that give the derivatives of $F$ and its dual, $\tilde{F}$. 