Introduction

- Till now we dealt only with finite elements having straight edges. In practical structures, the edges are always curved and to model such curved edges with straight edged elements will result in enormous increase in the degrees of freedom and the loss of accuracy.

- In addition, in many practical situations, it is always not required to have uniform mesh density throughout the problem domain. Meshes are always graded from fine (in the region of high stress gradient) to coarse (in the case of uniform stress field). These curved elements enable us to grade the mesh effectively.

- Isoparametric elements enable the modeling of curved domain with elements that are curved in nature.
Procedure

- The elements with curved boundaries are mapped to elements with the straight boundaries through a coordinate transformation that involves mapping functions, which are functions of the mapped coordinates.

- This mapping is established by expressing the coordinate variation (or transformation) as a polynomial of certain order and the order of the polynomial is decided by the number of nodes involved in the mapping.

- Since we would be working with the straight edged elements in the mapped domain, the displacement should also be expressed as a polynomial of certain order in the mapped coordinates.
Procedure (Cont)

• In this case, the order of the polynomial is dependent upon the number of degrees of freedom an element can support.
• Thus, we have two transformations, one involving the coordinates and the other involving the displacements.
• If the coordinate transformation is of lower order than the displacement transformation, then we call such transformation as \textit{sub-parametric transformation}. That is, if an element has \(n\) nodes, while all the \(n\) nodes participate in the displacement transformation, only few nodes will participate in the coordinate transformation.
• If the coordinate transformation is of higher order compared to the displacement transformation, such transformation is called \textit{super-parametric transformation}. In this case, only a small set of nodes will participate in the displacement transformation, while all the nodes will participate in the coordinate transformation.
Procedure (Cont)

- **In the FE formulation**, the most important transformation (as regards the FE formulation is concerned) is the one in which both the displacement and coordinate transformations are of same order, implying that all the nodes participate in both the transformations. Such a transformation is called the *iso-parametric transformation*.
**1-D Isoparametric Rod Element**

- Figure shows the 1-D rod element in the original rectangular coordinate system and the mapped coordinate system, with $\xi$ as the (1-D) mapped coordinate. Note that the two extreme ends of the rod, where axial degrees of freedom $u_1$ and $u_2$ are defined, the mapped coordinates are $\xi=-1$ and $\xi=1$, respectively. We now assume the displacement variation of the rod in the mapped coordinates as

$$u(\xi) = a_0 + a_1 \xi$$

- We now substitute $u(\xi = -1) = u_1$ and $u(\xi = 1) = u_2$ and eliminating the constants, we can write the displacement field in the mapped coordinates as

$$u(\xi) = \left(\frac{1-\xi}{2}\right)u_1 + \left(\frac{1+\xi}{2}\right)u_2 = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \left[N(\xi)\right]\{u\}$$
• We also assume that the rectangular \( x \) coordinate to vary with respect to mapped coordinate in the same manner of displacement. That is

\[
x = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[ N(\xi) \right] \{x\}
\]

• In the above equation, \( x_1 \) and \( x_2 \) are the coordinates of actual element in the rectangular \( x \) coordinate system. We can see that there is one-to-one correspondence of the coordinates in the original and the mapped system.
Stiffness matrix derivation

- The derivation of the stiffness matrix requires the computation of strain-displacement matrix $[B]$, which requires the evaluation of the derivatives of the shape functions with respect to original $x$ coordinate system.
- In the case of rod, there is only axial strain and hence $[B]$ matrix becomes

$$[B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

- The shape functions are functions of mapped coordinate $\xi$. Hence, derivative is first found in mapped coordinate and transformed to actual $x$ coordinate using coordinate transformation and *Jacobian*
That is, invoking the chain rule of the differentiation, we have

\[
\frac{dN_i}{dx} = \frac{dN_i}{d\xi} \frac{d\xi}{dx} \quad i=1,2 \quad (a)
\]

From the coordinate transformation, we have

\[
x = \frac{1 - \xi}{2} x_1 + \frac{1 + \xi}{2} x_2, \quad \frac{dx}{d\xi} = \frac{x_2 - x_1}{2} = \frac{L}{2} = J,
\]

\[
\frac{d\xi}{dx} = \frac{2}{L} = \frac{1}{J}, \quad dx = Jd\xi
\]

Using the above equations in Eqn (a), we see

\[
\frac{dN_i}{dx} = \frac{dN_i}{d\xi} \frac{1}{J} = \frac{dN_i}{d\xi} \frac{2}{L}
\]
• Substituting the shape functions, which is given by

\[ [N] = [N_1 \quad N_2] = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \]

we can obtain the shape function derivatives with respect to mapped coordinates and hence the \([B]\) matrix becomes

\[
\frac{dN_1}{d\xi} = -\frac{1}{2}, \quad \frac{dN_2}{d\xi} = \frac{1}{2}, \quad [B] = \frac{1}{J} \begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix}
\]

In the case of rod, there is only axial stress acting and as a result \([C]\), the material matrix for evaluating the stiffness matrix will have only \(E\), the Young’s modulus of the material. The stiffness matrix for a rod is given by

\[
[K] = \int\int [B]^T [C] [B] \, dV = \int\int\int [B]^T E[B] \, dA \, dx = \int\int\int [B]^T E[A][B] \, J \, d\xi
\]
Substituting in the above equation for $[B]$ matrix and Jacobian,, we get the stiffness matrix for a rod as

$$[K] = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Which is same as what was derived in the previous lecture

For lower order and straight edged elements, Jacobian is constant and not a function of mapped coordinate

For complex geometries and higher order elements, Jacobian is always a function of the mapped coordinate. In such cases, integration of the expression for computing the stiffness matrix will involve rational polynomials

To show this aspect, we will now derive the stiffness matrix for a higher order rod
Stiffness matrix for a Higher Order Rod

- The displacement variation for this element in the mapped coordinate is given by

\[ u(\xi) = a_0 + a_1\xi + a_2\xi^2 \quad (b) \]

- Following the same procedure as was done for the previous case, we first substitute

\[ u(\xi = -1) = u_1, \quad u(\xi = 0) = u_2, \quad u(\xi = 1) = u_3 \]

- This will give the following shape functions

\[ N_1 = \frac{\xi(-1 + \xi)}{2}, \quad N_2 = \left(1 - \xi^2\right), \quad N_3 = \frac{\xi(1 + \xi)}{2} \quad (c) \]

- Next, Jacobian requires to be computed for which we assume the coordinate transformation as

\[ x = \frac{\xi(-1 + \xi)}{2}x_1 + \left(1 - \xi^2\right)x_2 + \frac{\xi(1 + \xi)}{2}x_3 \quad (d) \]
• Taking the derivative with respect to the mapped coordinate, we get
\[ \frac{dx}{d\xi} = \frac{2\xi - 1}{2} x_1 - 2\xi x_2 + \frac{2\xi + 1}{2} x_3 = J, \quad dx = J d\xi \]

• Unlike in the 2-noded rod case, the Jacobian in the higher order rod case is a function of the mapped coordinate and its value changes as we move along the bar.

• If the coordinate \( x_2 \) coincide with the mid point of the rod, the value of the Jacobian becomes \( L/2 \).

• The \([B]\) matrix in this case becomes
\[
[B] = \frac{1}{J} \begin{bmatrix}
    \left( \frac{2\xi - 1}{2} \right) & -2\xi & \left( \frac{2\xi + 1}{2} \right)
\end{bmatrix}
\]
The stiffness matrix


\[ = \int_{-1}^{1} E A \frac{1}{J^2} \begin{bmatrix} \left( \frac{2\xi - 1}{2} \right) \\ -2\xi \\ \left( \frac{2\xi + 1}{2} \right) \end{bmatrix} \begin{bmatrix} \left( \frac{2\xi - 1}{2} \right) & -2\xi & \left( \frac{2\xi + 1}{2} \right) \end{bmatrix} J \, d\xi \]  \hspace{1cm} (e)

Obviously, the above expression cannot be integrated in closed form as we did earlier. It is in the form of rational polynomials, for which closed form solutions does not exist/ It has to be numerically integrated.
Numerical Integration and Gauss Quadrature

- Evaluation of stiffness and mass matrix, specifically for isoparametric elements involves expression such as the one given in Eqn (e), where the elements of the matrices are necessarily rational polynomials.
- Evaluation of these integrals in close forms is very difficult.
- Although there are different numerical schemes available, Gauss Quadrature is most ideally suited for isoparametric formulation as it evaluates the value of the integral between -1 to +1, which is the typical range of natural coordinates in isoparametric formulation.
Consider an integral of the form

\[ I = \int_{-1}^{+1} F d\xi, \quad F = F(\xi) = a_0 + a_1 \xi \]

- When this function requires to be integrated over a domain $-1 < \xi < 1$ with the length of the domain equal to 2 units.
- When the above expression is exactly integrated, we get the value of integral as $2a_0$.
- If the value of the integrand is evaluated at the mid point (that is, at $\frac{0}{2}=0$), and multiply with a weight $2.0$, we obtain the exact value of the integral.
- This result can be generalized for a function of any order as given by

\[ I = \int_{-1}^{+1} F d\xi \approx W_1 F_1 + W_2 F_2 + \ldots + W_n F_n \]
Hence, to obtain the approximate value of the integral $I$, we evaluate at several locations, multiply the resulting $F_i$ with the appropriate weights $W_i$ and add them together.

The points where the integrand is evaluated are called **sampling points**.

In Gauss Quadrature, these are the points of very high accuracy and sometimes referred to as **Barlow Points**.

These points are located symmetrically with respect to the center of the interval and symmetrically placed points have same weights.

The number of points required to integrate the integrand exactly depends on the degree of the highest polynomial involved in the expression.
• If $p$ is the highest degree of the polynomial in the integrand, then the minimum number of points $n$ required to integrate the integrand exactly is equal to $n=(p+1)/2$.

• That is for a polynomial of second degree, i.e. $p=2$, minimum number of points required to integrate is equal to 2.

• In the case of 2-D elements, the stiffness and mass matrix computation involves evaluation of the double integral of the form

$$I = \int_{-1}^{1} \int_{-1}^{1} F(\xi, \eta) \, d\xi \, d\eta = \int_{-1}^{1} \left[ \sum_{i=1}^{N} W_i F(\xi_i, \eta) \right] \, d\eta = \sum_{i=1}^{N} \sum_{j=1}^{M} W_i W_j F(\xi_i, \eta_j)$$

<table>
<thead>
<tr>
<th>Order $n$</th>
<th>Location $\xi_i$</th>
<th>Weight $w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$\pm 0.57735 \ 02691 \ 89626$</td>
<td>1.0</td>
</tr>
</tbody>
</table>
| 3         | $\pm 0.77459 \ 66692 \ 41483$  
0.00000 00000 00000 | 0.55555 55555 55556  
0.88888 88888 88889 |
| 4         | $\pm 0.86113 \ 63115 \ 94053$  
$\pm 0.33998 \ 20435 \ 84856$ | 0.34785 48451 37454  
0.65214 51548 62546 |
| 5         | $\pm 0.90617 \ 98459 \ 38664$  
$\pm 0.53846 \ 93101 \ 05683$  
0.00000 00000 00000 | 0.23692 68850 56189  
0.47862 86704 99366  
0.56888 88888 88889 |
Let us now revisit the stiffness matrix of the quadratic bar that was discussed earlier.

Writing Equation (e) again

\[ [K] = \int_{-1}^{1} EA \frac{1}{J^2} \left[ \begin{array}{ccc}
\frac{2\xi - 1}{2} & \frac{2\xi - 1}{2} & \frac{2\xi + 1}{2} \\
\frac{2\xi - 1}{2} & -2\xi & \frac{2\xi + 1}{2} \\
\frac{2\xi + 1}{2} & \frac{2\xi + 1}{2} & \frac{2\xi - 1}{2}
\end{array} \right] J d\xi \] (e)

From the above equation, the highest degree of mapped coordinate \( \xi \) is 2. Hence by the formula \( n = (p+1)/2 \), a 2 point Gauss integration is required.

From the table, the sampling points for integration will be located at \( \pm \frac{1}{\sqrt{3}} \) with weights being 1.0 for both points.
Using these in Equation (e), the stiffness matrix can be written as

$$[K] = \frac{2EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$
4-Noded 2-D Isoparametric FE Formulation

- Here, $x$-$y$ is the original coordinate system and $\xi$-$\eta$ is the mapped coordinate system.
- Each of the mapped coordinate ranges from $+1$ to $-1$.
- This element has 4 nodes and each node can support 2 degrees of freedom. In all the element has 8 degrees of freedom and the resulting stiffness matrix would be of size $8 \times 8$.

The displacement variation in the two coordinate directions ($u$ along $x$ direction and $v$ along $y$ direction) is given in terms of mapped coordinates as:

$$u(\xi, \eta) = a_0 + a_1\xi + a_2\eta + a_3\xi\eta$$
$$v(\xi, \eta) = b_0 + b_1\xi + b_2\eta + b_3\xi\eta$$
Substitution of the mapped coordinates at the four nodes, would result in the determination of shape functions. The displacement field as well of shape functions is given by

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix}
= \begin{bmatrix}
  N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\
  0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4
\end{bmatrix}
\begin{bmatrix}
  \{u\} \\
  \{v\}
\end{bmatrix}
= [N] \{u\}
\]

\[
\{u\} = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}^T
\]

\[
N_1 = \frac{(1 - \xi)(1 - \eta)}{4}, \quad N_2 = \frac{(1 + \xi)(1 - \eta)}{4},
\]

\[
N_3 = \frac{(1 + \xi)(1 + \eta)}{4}, \quad N_4 = \frac{(1 - \xi)(1 + \eta)}{4}
\]

The coordinate transformation between the original and mapped coordinates can be similarly written as

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= \begin{bmatrix}
  N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\
  0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4
\end{bmatrix}
\begin{bmatrix}
  \{x\}
\end{bmatrix}
= [N] \{x\}
\]

\[
\{x\} = \{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4\}
\]
• To compute the derivatives, we will invoke the chain rule. Noting that the original coordinates is a function of both mapped coordinate $\xi$ and $\eta$

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi}, \quad \frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \eta}$$

or

\[
\begin{pmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial \eta}
\end{pmatrix} = [J] \begin{pmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{pmatrix}
\]

• The numerical value of the Jacobian depends on the size, shape and the orientation of the element. Also,

\[
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial \eta}
\end{pmatrix} = [J]^{-1} \begin{pmatrix}
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \eta}
\end{pmatrix}
\]
• Using this equation, we can determine the derivatives required for the computation of \([B]\) matrix. Once this is done, we can derive the stiffness matrix for a plane element as

\[
[K] = t \int_{-1}^{1} \int_{-1}^{1} [B]^T [C][B] J \ d\xi \ d\eta
\]

• The above matrix should be numerically integrated with 2 point integration. The stiffness matrix will be 8 x 8. \([C]\) is the material matrix, and assuming plane stress condition, we have

\[
[C] = \frac{Y}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1 - \nu}{2}
\end{bmatrix}
\]
Some Numerical Examples

Analysis of a stepped Rod

- The aim of this example is to determine the stresses developed in this stepped bar due the central loading.

\[ k_1 = \frac{2EA}{L} \begin{bmatrix} u_1 & u_2 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \]

\[ k_2 = \frac{EA}{L} \begin{bmatrix} u_2 & u_3 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \]

Next the Matrix are assembled

\[ [K] = \frac{EA}{L} \begin{bmatrix} 2 & -2 & 0 \\ -2 & (2+1) & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \]
• Load and boundary conditions (BC) are \( u_1 = u_3 = 0 \),

• FE equation becomes,

\[
\begin{bmatrix}
2 & -2 & 0 \\
-2 & 3 & -1 \\
0 & -1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
u_2 \\
0 \\
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 \\
P \\
\end{bmatrix}
\]

\( F_2 = P \)

• Deleting the 1\textsuperscript{st} row and column, and the 3\textsuperscript{rd} row and column, we obtain,

\[
\frac{E A}{L} \begin{bmatrix}
3 \\
\end{bmatrix}\{u_2\} = \{P\}, \quad u_2 = \frac{P L}{3 E A}
\]

• Stress in element 1 is \( \sigma_1 = Y \varepsilon_1 = Y [B] \{u\}_{element1} \)

• The \([B]\) matrix derived earlier is used here and

\[
\{u\}_{element1} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T = \begin{bmatrix} 0 & u_2 \end{bmatrix}^T
\]

is the nodal vector of element 1. Using these in the above equation, we get stress in element 1 as
\[ \sigma_1 = Y \frac{u_2 - u_1}{L} = \frac{Y}{L} \left( \frac{PL}{3YA} - 0 \right) = \frac{P}{3A} \]

- Similarly, stress in element 2 is given by

\[ \sigma_2 = Y \varepsilon_2 = Y[B] \{u\}_{element2}, \quad \{u\}_{element2} = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}^T = \begin{pmatrix} u_2 \\ 0 \end{pmatrix}^T \]

Substituting the \([B]\) matrix and the displacement, the stress in element 2 is given by

\[ \sigma_2 = Y \frac{u_3 - u_2}{L} = \frac{Y}{L} \left( 0 - \frac{PL}{3YA} \right) = -\frac{P}{3A} \]
• **Observation:**

• In this case, the calculated stresses in elements 1 and 2 are exact within the linear theory for 1-D bar structures. It will not help if we further divide element 1 or 2 into smaller finite elements.

• For tapered bars, averaged values of the cross-sectional areas should be used for the elements.

• We need to find the displacements first in order to find the stresses, since we are using the *displacement based FEM.*
A Spring supported beam structure:

Here, following properties are given, $P=50.0 \text{ kN}$, $k=200 \text{ kN/m}$, $L=3.0 \text{ m}$ and $E=210 \text{ GPa}$, $I=2 \times 10^{-4} \text{ m}^4$.

- The beam structure will consist of middle support and the right end is supported by a spring. Such a configuration can be found in a MEMS structure on a silicon substrate, wherein the stiffness of the substrate is modeled as a spring.
- The beam structure is modeled with 2 elements, element 1 spanning nodes 1-2 and element 2 spanning nodes 2-3.
- Element stiffness of beam derived in Lecture 28 will be used here and we will use this to generate the elemental stiffness matrix for each element.
• As apposed to rods, beams support two degrees of freedom at each node, namely the transverse displacement \( w \) and the rotation \( \theta \).

• Next we will model the spring in the finite element framework. It is like a rod element with stiffness contributing to nodes 3 and 4 in the transverse \( z \) direction. The spring stiffness matrix is given by

\[
\begin{bmatrix}
  w_3 & w_4 \\
  k & -k \\
-k & k
\end{bmatrix}
\]

• As in the previous problem, we have to generate the stiffness matrix for each element and assemble. As before, the node 2 is common for both the elements and the global stiffness matrix will have contributions for both the elements for the transverse and rotation degrees of freedom corresponding to node 2.
Now adding all the stiffnesses, the global FE equation is given by

\[
\frac{EI}{L^3} \begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 & 0 \\
4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\
24 & 0 & -12 & 6L & 0 \\
8L^2 & -6L & 2L^2 & 0 \\
12 + k' & -6L & -k' & 4L^2 & 0 & 0 & k' \\
\end{bmatrix} \begin{bmatrix}
w_1 \\
\theta_1 \\
w_2 \\
\theta_2 \\
w_3 \\
\theta_3 \\
w_4 \\
\theta_4 \\
\end{bmatrix} = \begin{bmatrix}
V_1 \\
M_1 \\
V_2 \\
M_2 \\
V_3 \\
M_3 \\
V_4 \\
\end{bmatrix}
\]

where \( k' = \frac{L^3}{EI} k \). We now apply the boundary conditions

\begin{align*}
w_1 &= \theta_1 = w_2 = w_4 = 0 \\
M_2 &= M_3 = 0
\end{align*}

Deleting the first three and seventh \( V_3 = -P \) equations (rows and columns), we have the following reduced equation,

\[
\frac{YI}{L^3} \begin{bmatrix}
8L^2 & -6L & 2L^2 \\
-6L & 12 + k' & -6L \\
2L^2 & -6L & 4L^2 \\
\end{bmatrix} \begin{bmatrix}
\theta_2 \\
w_3 \\
\theta_3 \\
\end{bmatrix} = \begin{bmatrix}
o \\
-P \\
0 \\
\end{bmatrix}
\]
• Solving this equation, we obtain the deflection and rotations at node 2 and node 3, which is given by

\[
\begin{align*}
\begin{bmatrix}
\theta_2 \\
w_3 \\
\theta_3
\end{bmatrix}
&= -\frac{PL^2}{EI(12 + 7k')} \begin{bmatrix}
3 \\
7L \\
9
\end{bmatrix}
\end{align*}
\]

• The influence of the spring \( k \) is easily seen from this result. Plugging in the given numbers, we can calculate

\[
\begin{align*}
\begin{bmatrix}
\theta_2 \\
w_3 \\
\theta_3
\end{bmatrix}
&= \begin{bmatrix}
-0.002492 \text{rad} \\
-0.01744 \text{m} \\
-0.007475 \text{rad}
\end{bmatrix}
\end{align*}
\]

• From the global FE equation, we obtain the nodal reaction forces as,

\[
\begin{align*}
\begin{bmatrix}
V_1 \\
M_1 \\
V_2 \\
V_4
\end{bmatrix}
&= \begin{bmatrix}
-69.78 \text{kN} \\
-69.78 \text{kN} \cdot \text{m} \\
116.2 \text{kN} \\
3.488 \text{kN}
\end{bmatrix}
\end{align*}
\]
Summary

- In this lecture, we covered the following topics:
  1. Isoparametric formulation, its need and procedure
  2. Derived isoparametric formulation for elementary, and higher order rods and 2-D plane stress elements
  3. Some numerical examples involving finite elements were also solved
Thank You