Finite Element Method-Part I
Finite Element Equation Development
and Shape Functions

Lecture 28
Micro and Smart Systems
Finite Element procedure

1. It uses Weak form of the Governing Equation with the weight function $v$ is same as the dependent variable

2. The system under investigation is split into many sub-domains called elements and over each of these sub-domain, the dependent variable variation is assumed and converted in the form

$$\bar{u}(x, y, z, t) = \sum_{n=1}^{N} a_n(t) \phi_n(x, y, z)$$ (1)
FEM procedure

- This equation is the standard form for most of the approximate method that was described previously. However, here in FEM each of these have specific meaning
- $a_n(t)$ represents nodal degrees of freedom
- $\phi_n(x, y, z)$ represents shape function normally denoted as $N$
- The above variation of dependent variable, when substituted in the weak form of the governing equation and minimized as per PMPE OR HP, we get
  1. Set of algebraic Equation for Static problems
  2. Coupled set of ordinary differential equation for dynamic problems
FEM Procedure - Summary

1. The use of weak form of governing differential equation and assumption of the dependent variable variation over each element (Eqn. (1)) and its subsequent minimization to yield stiffness matrix and mass matrix (if the structures are subjected to inertial loads).

2. The size of these matrices depends on the number of nodes and the number of degrees of freedom each node can support.

3. Mass matrix formulated through the weak form of the equation is called the consistent mass matrix. There are other ways of formulating the mass matrix. That is, the total mass of the system can be distributed appropriately among all degrees of freedom. Such a mass matrix is diagonal and is called lumped mass matrix.

4. Damping matrix is normally not obtained through weak formulation. For linear system, they are obtained through linear combination of stiffness and mass matrix. Damping matrix obtained through such a procedure is called the proportional damping.
5. FEM comes under the category of stiffness method, where the dependent variable (say displacements in the case of structural systems) are the basic unknowns, the satisfaction of compatibility of displacements across the element boundaries is automatic as we begin the analysis with displacement assumption.

6. Equilibrium of the forces are ensured only within the element. Global Equilibrium is not ensured. It is accomplished by assembly of the elemental matrices that are sharing the common interfaces.

7. Similarly, the force vector acting on each node, are assembled to obtain global force vector. If the load is distributed on a segment of the complex domain, then using equivalent energy concept, it is split into concentrated loads acting on the respective nodes that make up the segment. The size of assembled stiffness, mass and damping matrices is equal to $n \times n$, where $n$ is the total number degrees of freedom in the discretized domain.
8. After assembly of matrices, the displacement boundary conditions are enforced, which could be homogenous or non-homogenous. If the boundary conditions are homogenous, then, the corresponding rows and columns are eliminated to get the reduced stiffness, mass and damping matrices. In the case of static analysis, the obtained matrix equation involving stiffness matrix, is solved to obtain the nodal displacements. In the case of dynamic analysis, we get a coupled set of ordinary differential equation, which is solved by either modal methods or time marching scheme
Start

Read mesh, constraints and loads. Enter material model – D.

Pre-compute all stationary element properties

Initialize Displacements $u_0, u, -u$

Begin Loop over Elements

Begin Loop

Update Displacements $u_{t+\Delta t} = u_t, -u_t$

Strain $\varepsilon$

Stress $\sigma$

Element Nodal Forces $F^k$

Net Nodal Forces $F = \sum F^k$

Apply Constraints & Loads for $u_{t+\Delta t}$

New Displacements $u_{t+\Delta t}$

Equation of Motion

$\sum F = M\ddot{u}$
Finite Element Equations

\[ \delta \int_{t_1}^{t_2} (T - U + W_{nc}) dt = 0 \]  \hspace{1cm} (8.69)

where the kinetic energy \( T \) is given by

\[ T = \frac{1}{2} \int_V \rho \left( a^2 + \ddot{v}^2 + \ddot{w}^2 \right) dV \]

Taking the first variation and integrating, we get

\[ \int_{t_1}^{t_2} \delta \mathbf{T} \, dt = \int_{t_1}^{t_2} \mathbf{\rho} \left( \frac{du}{dt} \frac{d(\delta u)}{dt} + \frac{dv}{dt} \frac{d(\delta v)}{dt} + \frac{dw}{dt} \frac{d(\delta w)}{dt} \right) dV \, dt \]

Integrating by parts, the above equation and noting that the first variation vanishes at times \( t_1 \) and \( t_2 \)

\[ \int_{t_1}^{t_2} \delta \mathbf{T} \, dt = -\int_{t_1}^{t_2} \mathbf{\rho} \left( \ddot{u} \delta u + \ddot{v} \delta v + \ddot{w} \delta w \right) dV \, dt = -\int_{t_1}^{t_2} \mathbf{\rho} \{\delta \mathbf{d}\}^T \{\ddot{d}\} dV \, dt \] \hspace{1cm} (8.70)

where, \( \{\ddot{d}\} = \{\ddot{u} \, \ddot{v} \, \ddot{w}\}^T \) represent the acceleration vector and \( \{\delta \mathbf{d}\} = \{\delta u \, \delta v \, \delta w\}^T \) represent the vector containing the first variation of displacements.
The strain energy for a 3-D body in terms of stresses and strains is given by

$$U = \frac{1}{2} \int \left( \sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{zz} \varepsilon_{zz} + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx} \right) dV = \frac{1}{2} \int \{\varepsilon\}^T \{\sigma\} dV \quad (8.71)$$

For linear elastic case, the constitutive law by Equation (8.71) can be written as $\{\sigma\} = [C] \{\varepsilon\}$. Hence the strain energy becomes

$$U = \frac{1}{2} \int \{\varepsilon\}^T [C] \{\varepsilon\} dV$$

$$\int_{t_1}^{t_2} \delta U \, dt = \int_{t_1}^{t_2} \int \{\delta \varepsilon\}^T [C] \{\varepsilon\} dV \, dt \quad (8.72)$$

The work done by the body forces, surface forces, damping elements and the concentrated forces are clubbed under $W_{nc}$. That is $W_{nc} = W_B + W_S + W_D$. Work done by the body forces is given by

$$W_B = \int (B_x u + B_y v + B_z w) \, dV = \int \{d\}^T \{B\} \, dV \quad (8.73)$$

The first variation of the body force work is given by

$$\int_{t_1}^{t_2} \delta W_B \, dt = \int_{t_1}^{t_2} (B_x \delta u + B_y \delta v + B_z \delta w) \, dV \, dt = \int_{t_1}^{t_2} \int \{\delta d\}^T \{B\} \, dV \, dt \quad (8.74)$$
The first variation of this work is given by

\[ \int_{t_1}^{t_2} \delta W_S \, dt = \int_{t_1}^{t_2} \left\{ \delta \{ \mathbf{d} \} \right\}^T \{ t_s \} \, dS \, dt \]  
(8.75)

Similarly, the first variation of the work done by the damping force is given by

\[ \int_{t_1}^{t_2} \delta W_D \, dt = -\int_{t_1}^{t_2} \left\{ \delta \{ \mathbf{d} \} \right\}^T \{ F_D \} \, dV \, dt \]  
(8.76)

If the damping is viscous type, then the damping force is proportional to the velocity and is given by \( \{ F_D \} = \eta \{ \dot{\mathbf{d}} \} \), where \( \eta \) is the damping coefficient and \( \{ \dot{\mathbf{d}} \} = \{ \dot{u} \quad \dot{v} \quad \dot{w} \}^T \) is the velocity vector in the three coordinate directions. Now using Equations (8.70), (8.72), (8.73), (8.75) and (8.76) in the Hamilton’s Principle (Equation (8.69)), we get

\[ -\int_{t_1}^{t_2} \int_{t_1}^{t_2} \left\{ \delta \{ \mathbf{d} \} \right\}^T \rho \{ \ddot{\mathbf{d}} \} \, dV \, dt - \int_{t_1}^{t_2} \int_{t_1}^{t_2} \left\{ \delta \{ \mathbf{d} \} \right\}^T \left[ C \right] \{ \mathbf{e} \} \, dV \, dt + \int_{t_1}^{t_2} \int_{t_1}^{t_2} \left\{ \delta \{ \mathbf{d} \} \right\}^T \left\{ B \right\} \, dV \, dt + \int_{t_1}^{t_2} \int_{t_1}^{t_2} \left\{ \delta \{ \mathbf{d} \} \right\}^T \{ t_s \} \, dS \, dt - \int_{t_1}^{t_2} \int_{t_1}^{t_2} \left\{ \delta \{ \mathbf{d} \} \right\}^T \{ F_D \} \, dV \, dt = 0 \]  
(8.77)
\[ \{d(x, y, z, t)\} = [N(x, y, z)]\{u_e(t)\} \]

\[ \{\ddot{d}\} = [N]\{\dot{u}_e\}, \quad \{\dddot{d}\} = [N]\{\ddot{u}_e\} \quad \text{and} \quad \{\delta d\} = [N]\{\delta u_e\} \]

\[
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z}
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
= \{\delta u_e\}^T \left[ \int \rho [N]^T [N] dV \right] \{\ddot{u}_e\} = \{\delta u_e\}^T [M] \{\ddot{u}_e\} \quad (8.83)
\]

\[
\int \{\delta \varepsilon\}^T [C] \{\varepsilon\} dV = \int \{\delta u_e\}^T [B]^T [C][B] \{u_e\} dV = \{\delta u_e\}^T \left[ \int [B]^T [C][B] dV \right] \{u_e\}
\]

\[
= \{\delta u_e\}^T [K] \{u_e\} 
\]
\[
\int \delta d^T B dV = \int \delta u^T N^T B dV \\
= \delta u^T \left[ \int N^T B dV \right] = \delta u^T f_B
\]

\[
\int \delta d^T t_s dS = \delta u^T f_s, \quad f_s = \int N^T t_s dS
\]

\[
\int_{t_1}^{t_2} \delta u_e^T \left[ M \ddot u_e + D \dot u_e + K u_e - f_B - f_s \right] dt = 0
\]

\[
- \int \delta d^T \eta d dV = -\delta u_e^T \left[ \int \eta N^T N dV \right] \dot u_e = -\delta u_e^T D \ddot u_e
\]

\[
[M \dddot u_e + D \ddot u_e + K u_e] = \{R\}
\]
Finite Element Terminologies

- **Degree of freedom**: It is the number of independent motions a structure can support. Example, Rod can support only axial motions and hence each node can have only one degree of freedom, namely axial motion.

A beam can support both transverse motion and rotation, that is each node in a beam will have 2 degrees of freedom.

A composite plate can support, 3 translational motion in 3 coordinate direction and 2 rotations in the in-plane directions. Hence a plate node can have 5 degrees of freedom. If twist is present, one can add a sixth degree of freedom by adding a rotation about the axial direction.
\[ \dot{\theta} = \frac{d\omega}{dx} \]
Finite Element Terminologies (cont)

- **Continuity:** Across the elemental boundary, all the displacement needs to be continuous. For example, in the case of rods or plane stress elements, it is necessary that only the displacements be continuous. Such continuity requirement is called \( C^0 \) continuous elements.

In the case of beams and plates, which has slope d.o.f and when the slope is derived from displacements \( \theta \), in this case, both the dependent variable and its first derivative needs to be continuous. Such continuity requirement is called \( C^1 \) continuous elements. However, if the slopes are independently interpolated (as in the case of Timoshenko beam or Mindlin plate), then maintaining \( C^0 \) continuity is sufficient.
Shape Functions
Introduction

• Shape functions describe the variation of a dependent variable, say displacement within a element. They are normally expressed in the form

\[ \bar{u}(x, y, z, t) = \sum_{n=1}^{N} a_n(t) \phi_n(x, y, z) \]

• In the above expression \( \phi_n(x, y, z) \) normally referred to as shape function.

• These are constructed using polynomial approximations and the order of which is dictated by the number of degrees of freedom an element can support.
A rod element can carry only axial motion in the x-direction. Hence, it has 2 axial deformation as degrees of freedom at x=0 and x=L. If we assume polynomial as assumed variation of deformation, it can only be a linear polynomial. Hence, the assumed variation can be written as

\[ u(x,t) = a_0(t) + a_1(t)x \]  \hspace{1cm} (1)

In the above equation, note that the constants \(a_0\) and \(a_1\) are time dependent if the problem is dynamic in nature.

We now substitute \(u(x=0) = u_1\) and \(u(x=L) = u_1\) in the above equation. This will enable us to write the constants \(a_0\) and \(a_1\) in terms of the nodal displacements \(a_0\) and \(a_1\).
• That is

\[
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} a_0 \\
    a_1
\end{bmatrix}
\]

inverting

\[
\begin{bmatrix}
    a_0 \\
    a_1
\end{bmatrix} = \frac{1}{L} \begin{bmatrix} L & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix}
\]

• Substituting for the constants in Eqn. (1), we get

\[
u(x) = \left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2
\]

(2)

• This can be written as

\[
u(x, t) = \sum_{i=1}^{2} N_i u_i \quad \text{or} \quad \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\
    u_2
\end{bmatrix},
\]

\[
N_1 = \left(1 - \frac{x}{L}\right), \quad N_2 = \left(\frac{x}{L}\right)
\]

• Here \(N_1\) and \(N_2\) are the shape functions for a rod element.
Properties of Shape functions

- $N_1$ takes a value of 1 at node 1 and zero at node 2 and $N_2$ takes a value of 1 at node 2 and zero at node 1
- Let is evaluate $N_1 + N_2 = 1 - \frac{x}{L} + \frac{x}{L} = 1$
- That is sum of shape function is always one
Beam Element

- A beam element shown in has two nodes and each node has two degrees of freedom, namely the transverse displacement \( w \) and rotation \( \theta = \frac{dw}{dx} \).
- The nodal degrees of freedom vector is given by
  \[
  \{u\} = \begin{bmatrix} w_1 & \theta_1 & w_2 & \theta_2 \end{bmatrix}^T
  \]
- Require \( C^1 \) continuity requirement to be satisfied.
- To support four degrees of freedom, we need a cubic polynomial for transverse displacement. That is
  \[
  w(x,t) = a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3 \quad (2)
  \]
Now, we substitute

\[ w(0, t) = w_1(t), \quad \theta(0, t) = dw(0, t) / dx = \theta_1(t), \]
\[ w(L, t) = w_2(t), \quad \text{and} \quad \theta(L, t) = dw(L, t) / dx = \theta_2(t) \]

Inverting the above matrix, we can write the unknown coefficients as \( \{a\} = [G]^{-1} \{u\} \)

Substituting these coefficients in Equation (2) we can write the transverse displacements as

\[ w(x, t) = \left[ N_1(x) \quad N_2(x) \quad N_3(x) \quad N_4(x) \right] \{u(t)\}, \]

\[ N_1(x) = 1 - 3 \left( \frac{x}{L} \right)^2 + 2 \left( \frac{x}{L} \right)^3, \quad N_2(x) = x \left( 1 - \frac{x}{L} \right)^2 \]
\[ N_3(x) = 3 \left( \frac{x}{L} \right)^2 - 2 \left( \frac{x}{L} \right)^3, \quad N_4(x) = x \left[ \left( \frac{x}{L} \right)^2 - \left( \frac{x}{L} \right) \right] \]
• These shape functions satisfy all the properties of the shape functions

• Note that , \( N_2 \), which corresponds to shape function of the slope at Node 1 does not take the value of unity at \( x=0 \). However, \( dN_2/dx \) is equal to 1 at \( x=0 \). Similarly \( dN_4/dx \) and not \( N_4 \), that takes the value of unity at \( x=L \)

Note that the cubic interpolation functions are derived by interpolating \( w \) as well as its derivative \( dw/dx \) at the nodes. Such polynomials are known as the Hermite family of interpolation functions, and \( \phi^e_i \) are called the Hermite cubic interpolation (or cubic spline) functions.
Let us now consider a rectangular finite element of length $2a$ and width $2b$.

This element has four nodes and each node can support two degrees of freedom, (namely) the two displacements, $u(x,y)$, and $v(x,y)$ in the two coordinate directions. Since there are 4 nodes, we can assume the interpolating polynomial as

\[
\begin{align*}
  u(x,y) &= a_0 + a_1 x + a_2 y + a_3 xy \\
  v(x,y) &= b_0 + b_1 x + b_2 y + b_3 xy
\end{align*}
\]
In the above equation, we substitute

\[ u(-a,b) = u_2, \quad v(-a,b) = v_2, \quad u(-a,-b) = u_1, \quad v(-a,-b) = v_1, \quad u(a,-b) = u_4, \quad v(a,-b) = v_4 \]

These help us to relate the nodal displacements to the unknown coefficients as

\[ \{u\} = \{C\} \{\{a\}\} \]

Inverting (the above relation) and substituting for unknown coefficients, we can write the displacement field and the shape functions as

\[
\begin{align*}
    u(x,y) &= [N]\{u\} = [N_1(x,y) \quad N_2(x,y) \quad N_3(x,y) \quad N_4(x,y)]\{u\} \\
    v(x,y) &= [N]\{v\} = [N_1(x,y) \quad N_2(x,y) \quad N_3(x,y) \quad N_4(x,y)]\{v\} \\
    \{u\} &= \{u_1 \quad u_2 \quad u_3 \quad u_4\}^T, \quad \{v\} = \{v_1 \quad v_2 \quad v_3 \quad v_4\}^T \\
    N_1(x,y) &= \frac{(x-a)(y-b)}{4 \, ab}, \quad N_2(x,y) = \frac{(x-a)(y+b)}{4 \, ab}, \\
    N_3(x,y) &= \frac{(x+a)(y+b)}{4 \, ab}, \quad N_4(x,y) = \frac{(x+a)(y-b)}{4 \, ab} \end{align*}
\]

These shape functions satisfy all the properties.
Triangular Element

Deriving the shape functions through conventional means for a triangle is very cumbersome. Here, we will use area coordinates.

Consider a triangle having coordinates of the three vertices as $\begin{pmatrix} x_1, y_1 \\ x_2, y_2 \end{pmatrix}$ and $\begin{pmatrix} x_3, y_3 \end{pmatrix}$.

Consider an arbitrary point $P$ inside the triangle. This point will split the triangle into three smaller triangles of area $A_1$, $A_2$, and $A_3$, respectively. Let $A$ be the total area of the triangle, which can be written in terms of nodal coordinates as

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$
• We will define the area coordinates for the triangle as

\[ L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A} \]

• Thus the position of point \( P \) is (thus) given by coordinates \( x \). These coordinates, which are normally referred to as area coordinates, are not independent and they satisfy the relation

\[ L_1 + L_2 + L_3 = 1 \]

• These area coordinates are related to the global \( x-y \) coordinate system through

\[ x = L_1 x_1 + L_2 x_2 + L_3 x_3, \quad y = L_1 y_1 + L_2 y_2 + L_3 y_3 \]

where \( L_i = \frac{(a_i + b_i x + c_i y)}{2A} \) \( i = 1, 2, 3 \) and

\[ a_1 = x_2 y_3 - x_3 y_2, \quad b_1 = y_2 - y_3, \quad c_1 = x_3 - x_2 \]
• The other coefficients are obtained by cyclic permutation.
• The above equation requires to be used when the derivative with respect to the actual coordinates \(x\) and \(y\) are required.
• Now, one can write the shape functions for the triangle as

\[
\begin{align*}
  u &= N_1 u_1 + N_2 u_2 + N_3 u_3 \\
  v &= N_1 v_1 + N_2 v_2 + N_3 v_3 \\
  N_1 &= L_1, \quad N_2 = L_2, \quad N_3 = L_3
\end{align*}
\]
• These shape functions also follow the normal rules. That is at point \(A\) where the value of \(L_1=1\), the shape functions take the value of 1. At the same point, \(L_2=L_3=0\). Similarly, at the other two vertices, \(L_2\) and \(L_3\) take a unit value, while the other two goes to zero.
Rules for choosing Interpolation Functions

1. The assumed solution should be able to capture the rigid body motion. This can be made sure by retaining a constant part in the assumed solution.

   \[ u = \bar{u}_0 + u_1 + u_2 x + \ldots \]

2. The assumed solution must be able to attain the constant strain rate as the mesh is refined. This can be assured by retaining the linear part of the assumed function in the interpolating polynomial.
4. Most second order systems require only $C^0$ continuity, which are easily met in most FE formulation. However, for higher order systems such as Bernoulli-Euler beams or elementary plates, one requires $C^1$ continuity, which are extremely difficult to satisfy, especially for plate problems, where inter-element slope continuity is very difficult to satisfy. In such situations, one can use shear deformable models, that is, models that also includes the effect of shear deformations. In such models, slopes are not derived from the displacements and are independently interpolated. This relaxes the $C^0$ continuity requirement. However, when such elements are used in thin beam or plate models, where the effect of shear deformations are negligible, the displacements predicted would be many orders smaller than the correct displacements. Such problems are called the *shear locking* problems.
5. The order of assumed interpolating polynomial is dictated by the highest order of the derivative appearing in the energy functional. That is, the assumed polynomial should be at least one order higher than that is appearing in the energy functional.

\[
\text{Beam } U = \int_0^L \sum_{n} E I \left( \frac{d^2 \omega}{dx^2} \right) dx
\]

- In summary, for all the elements we can express the displacements in terms of shape functions and the nodal displacements as

\[
u = \sum_{n=1}^{N} N_n \text{ This spatial discretization will be used in the weak form of the governing equation to obtain the FE governing equation}
\]
Finite Element Formulation

• Rod Element Formulation:

A rod can carry one dof/node and 2 dof per element. The elemental variation of deformation in terms of shape function is given by

\[ u(x,t) = \sum_{i=1}^{2} N_i u_i \quad \text{or} \quad \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \]

\[ N_1 = \left(1 - \frac{x}{L}\right), \quad N_2 = \left(\frac{x}{L}\right) \]

We use the shape function information to derive the stiffness and mass matrix, which is given by

\[ [K] = \int_{V} [B]^T [C] [B] dV, \quad [M] = \int_{V} \rho [N]^T [N] dV \]
Rod Element (Cont)

- \([B]\) is the strain displacement matrix. Only relevant strain is the axial strain \(\varepsilon_{xx}\) corresponding to axial stress \((\sigma_{xx})\) and in \([C]\), only \(E\) is relevant.
- Hence \(\varepsilon_{xx} = \frac{du}{dx} = \frac{dN}{dx} \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\)

\[
\begin{bmatrix}
\frac{dN_1}{dx} \\
\frac{dN_2}{dx}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{L} & \frac{1}{L}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]

Or

\(\varepsilon_{xx} = [B] \{u\}\)

\[
[K] = \int_V E [B]^T [B] dV = \int_0^L E [B]^T [B] dx \int_A dA
\]

\[
= \int_0^L EA \begin{bmatrix}
-1 \\
\frac{1}{L} \\
\frac{1}{L}
\end{bmatrix} \begin{bmatrix}
-1 & 1 \\
\frac{1}{L} & \frac{1}{L}
\end{bmatrix} dx = \frac{EA}{L} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\]
Rod Element-Mass Matrix

- Mass matrix is given by

\[
[M] = \int_V \rho [N]^T [N] \, dV = \int_0^L \rho [N]^T [N] \, dx \int_A dA
\]

\[
= \int_0^L \rho A [N]^T [N] \, dx = \int_0^L \rho A \left[ \begin{array}{c} \left(1 - \frac{x}{L}\right) \\ \frac{x}{L} \end{array} \right] \left[ \begin{array}{cc} 1 - \frac{x}{L} & \frac{x}{L} \end{array} \right] \, dx
\]

\[
[M] = \frac{\rho AL}{6} \left[ \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right]
\]
Beam Element Formulation

- The Shape function for beam was derived earlier. We use this here. We first derive the strain displacement matrix \([B]\). The displacement field for the beam is given by

\[ u(x, y, z, t) = -z \frac{dw}{dx} , \quad w(x, y, t) = w(x, t) \]

- The relevant strains are

\[ \varepsilon_{xx} = \frac{du}{dx} = -z \frac{d^2w}{dx^2} , \quad \varepsilon_{zz} = \frac{dw}{dz} = 0 , \quad \gamma_{xy} = \frac{dw}{dx} + \frac{dy}{dz} = \frac{dw}{dx} - \frac{dw}{dx} = 0 \]

- Hence, \( \varepsilon_{xx} \) is the relevant strain and correspondingly \( \sigma_{xx} \) is the only relevant stress and hence the matrix \([C]\) is equal to \( E \).
Beam Element Formulation

- Hence Strain displacement matrix is obtained as follows

\[
\varepsilon_{xx} = \frac{du}{dx} = -z \frac{d^2}{dx^2} \left[ \begin{array}{cccc} N_1 & N_2 & N_3 & N_4 \end{array} \right] \left\{ \begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array} \right\}
\]

\[
= -z \left[ \begin{array}{cccc} \frac{d^2 N_1}{dx^2} & \frac{d^2 N_2}{dx^2} & \frac{d^2 N_3}{dx^2} & \frac{d^2 N_4}{dx^2} \end{array} \right] \left\{ \begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array} \right\} = -z[B]\{u\}
\]

- The stiffness matrix is given by

\[
[K] = \int_V Ez^2[B]^T[B]dV = \int_0^L E[B]^T[B]dx \int z^2dA
\]

\[
= EI \int_0^L [B]^T[B]dx = \frac{EI}{L^3} \begin{bmatrix}
12 & -6L & -12 & -6L \\
-6L & 4L^2 & 6L & 2L^2 \\
-12 & 6L & 12 & -6L \\
-6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\]
### Beam Formulation: Mass Matrix

- Mass matrix is given by

\[
[M] = \int_V \rho [N]^T [N] dV = \int_0^L \rho [N]^T [N] dx \int dA
\]

\[
= \int_0^L \rho A [N]^T [N] dx = \int_0^L \rho A \begin{bmatrix}
N_1^2 & N_1 N_2 & N_1 N_3 & N_1 N_4 \\
N_1 N_2 & N_2^2 & N_2 N_3 & N_2 N_4 \\
N_1 N_3 & N_2 N_3 & N_3^2 & N_3 N_4 \\
N_1 N_4 & N_2 N_4 & N_3 N_4 & N_4^2
\end{bmatrix} dx
\]

\[
= \frac{\rho AL}{420} \begin{bmatrix}
156 & -22L & 54 & 13L \\
-22L & 4L^2 & -13L & -3L^2 \\
54 & -13L & 156 & 22L \\
13L & -3L^2 & 22L & 4L^2
\end{bmatrix}
\]

\[
\]
Few Observations

- Both Mass and Stiffness matrices are symmetric.
- When the stiffness and mass matrices are assembled, they are also symmetric and banded. That is all elements of the stiffness matrix need not be stored. The bandwidth is computed from the formula:
  \[ B = (N_D + 1) \times N_{dof} \]
Many a times masses are lumped. That is the total mass of the structures are lumped corresponding to translational d.o.f, while the total mass moment of inertia is equally lumped to all rotational d.o.f.

The distributed loads are converted to equivalent concentrated loads on the nodes that support this distributed load using the formula

\[
\{f\} = \int_s [N]^T q(x, y) ds
\]
Summary

- In this lecture, we studied the following
  1. We developed the FE equation
  2. Outlined the procedure for shape function determination and derived them for few elements
  3. Outlined the procedure element formulation for rods and beams. Similar procedure can be adopted for the formulation of other elements
  4. We discussed the few issues concerning the choice of interpolating functions, storage etc.