Fluid Mechanics

Q1.
(a) For a steady two-dimensional incompressible flow through a nozzle, the velocity field is given by \( \vec{V} = u_0(1 + 2x/L) \hat{i} \), where \( x \) is the distance along the axis of the nozzle from its inlet plane and \( L \) is the length of the nozzle. Find
(i) an expression of the acceleration of a particle flowing through the nozzle, and
(ii) the time required for a fluid particle to travel from the inlet to the exit of the nozzle.
(b) An open rectangular tank of 5 m x 4 m x 4 m is 3 m high. It contains water up to a height of 2 m and is accelerated horizontally along the longer side. Determine the maximum acceleration that can be given without spilling the water and also calculate the percentage of water spilt over, if this acceleration is increased by 20%. Derive the formula used for the solution of the problem.
(c) A single uniform pipe joins two reservoirs. Calculate the percentage increase of flow rate obtainable if, from the midpoint of this pipe, another of the same diameter is added in parallel to it. Neglect all losses except pipe friction and assume a constant and equal \( f \) for both pipes.

[8 + 10 + 7 = 25 Marks]

Q2.
(a) Calculate the thrust required to run a motor-boat 5 m long at 100 m/s in a lake if the force required to tow a 1:30 model in a reservoir is 5 N. Neglect the viscous resistance due to water in comparison to the wave making resistance. Derive the dimensionless parameters used for the solution of the problem.
(b) A tank of area \( A_0 \) is draining in laminar flow through a pipe of diameter \( D \) and length \( L \), as shown in the figure.
(i) Neglecting the exit-jet kinetic energy and assuming the pipe flow is driven by the hydrostatic pressure at its entrance, derive a formula for the tank level \( h(t) \) if its initial level is \( h_0 \), assuming the pipe flow to be laminar.
(ii) Repeat the above derivation without neglecting the exit-jet kinetic energy and assuming the pipe flow to be highly turbulent.
(c) Given the stress tensor components for a 2-D flow-field as
\[
\begin{bmatrix}
a/2 & 0 \\
0 & -3a/2
\end{bmatrix}
\]
(i) Split the stress tensor into isotropic and anisotropic parts.
(ii) What are the physical origins of these two parts?
(iii) What are the principal directions and values of the principal stress? How are those related to pressure and viscous stresses?

[9 + 10 + 6 = 25 Marks]

Q3.
(a) Two viscous, incompressible, immiscible fluids of same density \( = \rho \) but different viscosities (viscosity of the lower fluid layer = \( \mu_1 \) and that of the upper fluid layer = \( \mu_2 < \mu_1 \) ) flow in separate layers between parallel boundaries located at \( y = \pm H \). Thickness of each fluid layer is identical and their interface is flat. The flow is driven by a constant favourable pressure gradient of \( \frac{dp}{dx} \). Derive expressions for the velocity profiles in the fluid layers. Also, qualitatively make a sketch of the velocity profiles. Assume the flow to be steady and the plates to be of infinitely large width.

(b) Assume that flow around a circular cylinder separates at \( \theta = 120^\circ \). Till the point of separation, the pressure distribution closely follows the potential flow theory, with \( C_p = \frac{p_{\infty} - p}{1/2 \rho u_{\infty}^2} = 1 - 4 \sin^2 \theta \). Beyond the separation point, the pressure in the wake region remains approximately constant. Neglecting the skin frictional drag on the cylinder, determine the drag coefficient.

Q4.
(a) The free stream velocity for a flow is expressed in terms of the stream-wise boundary layer coordinate as \( U_\infty = cx^m \), where \( m \) is a constant. Write the boundary layer equations for this case in the most simplified form.

(b) Consider growth of a boundary layer for flow over plate of length \( L \). At \( x = x_c \) \( (x_c < L) \), flow undergoes a transition towards turbulence. Qualitatively sketch the variation of wall shear stress as a function of axial position along the plate over the entire length.

(c) The velocity profile within boundary layer for steady, two-dimensional, constant density, laminar flow over a flat plate is given as:
\[
\frac{u}{u_\infty} = a_0 + a_1 \frac{y}{\delta}
\]
Using suitable boundary conditions, evaluate the constants \( a_0 \) and \( a_1 \). Using the above velocity profile, evaluate the total shear force on a plate of length \( L \) and width \( w \), and the drag coefficient \( C_D \).

(d) A flat plate is exposed to a fluid flow with a free stream parallel to the axis of the plate. In another case, this plate is replaced by another plate of half the length of the previous plate, all other conditions remaining unaltered. In both the cases, flow over the entire length of the plate is laminar. With an order of magnitude analysis, determine the ratio of the drag coefficients for these two cases.

(e) Consider the growth of boundary layer for flow over a surface with favourable pressure gradient. Qualitatively sketch the plots of \( u, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2} \), as a function of \( y \), where \( y \) is the boundary layer coordinate.
1.
(a)
(i) The velocity component is given as 
\[ u = u_0 \left(1 + 2x/L\right) \]
Hence, 
\[ \frac{\partial u}{\partial x} = \frac{2u_0}{L} \]
For the given velocity field, acceleration can be written as 
\[ a = u \frac{\partial u}{\partial x} \]
\[ a = u_0 \left(1 + 2x/L\right) \frac{2u_0}{L} = \frac{2u_0^2}{L} \left(1 + 2x/L\right) \]
(ii)
Let \( t \) be the time required for a fluid particle to travel from the inlet to the exit of the nozzle.
From the given velocity field, we have 
\[ V = \frac{dx}{dt} = u_0 \left(1 + 2x/L\right) \]
or 
\[ \frac{dx}{1 + 2x/L} = u_0 dt \]
Integrating the above equation, we obtain 
\[ \int_0^L \frac{dx}{1 + 2x/L} = \int_0^t u_0 dt \]
which gives 
\[ t = \frac{L}{2u_0} \ln 3 \]
(b)
For the maximum acceleration without spilling the water, the free surface takes the shape as shown in figure below.

From the geometry of the figure above, we have 
\[ \tan \theta = \frac{a_{x, \text{max}}}{g} \]
This is the maximum acceleration that can be given without spilling the water.

Now the acceleration is increased by 20%. Therefore, the new acceleration is

\[ a_x = 3.924 \times 1.2 = 4.7088 \text{ m/s}^2 \]

Now,

\[ \tan \theta = \frac{h}{L} = \frac{a_x}{g} \]

or,

\[ h = \frac{4.7088}{9.81} \]

or,

\[ h = 2.4 \text{ m} \]

The new configuration is shown in the figure below.

Volume of water left in the tank is

\[ = \left[ \frac{1}{2} \times 2.4 \times 5 + 0.6 \times 5 \right] \times 4 \]

\[ = 36 \text{ m}^3 \]

Initial volume of water in the tank is

\[ = 5 \times 4 \times 2 = 40 \text{ m}^3 \]

Percentage of water spilt over is

\[ \frac{40 - 36}{40} \times 100 = 10\% \]

(c)

Let the diameter of the pipe be \( D \).

Case 1:

When the single pipe joins two reservoirs, as shown in the figure below, the loss of head is

\[ h_{f1} = f \frac{L V^2}{D 2g} \]

where is \( V \) the average velocity of fluid in the pipe.

For this case, the discharge is given by

\[ Q = AV \]
Case 2:
When another pipe is added in parallel to the main pipe from the midpoint as shown in the figure below, the loss of head is

\[ h_{f2} = f \frac{L}{D} \frac{V_2^2}{2g} + f \frac{L}{D} \frac{V_1^2}{2g} \]  \hspace{1cm} (1)

From continuity equation, we have

\[ Q_1 = Q_2 + Q_3 \]

Since the diameter and length of both the parallel pipes are same, we have

\[ Q_2 = Q_3 \]

\[ \therefore \quad Q_2 = Q_3 = \frac{Q}{2} \]

or

\[ V_2 = V_3 = \frac{V}{2} \]

Substituting the value of \( V_2 \) in Eq.(1), we get

\[ h_{f2} = f \frac{L}{D} \frac{V^2}{2g} + f \frac{L}{D} \frac{4V^2}{8g} = 5 \quad f \frac{L}{D} \frac{V^2}{2g} \]

Equating the head losses, we have

\[ h_{h} = h_{f2} \]

\[ f \frac{L}{D} \frac{V^2}{2g} = 5 \quad f \frac{L}{D} \frac{V^1}{2g} \]

or

\[ V_1 = 1.26V \]

For this case, the discharge is given by

\[ Q_1 = AV_1' = 1.26AV \]

Therefore, the percentage increase in the flow rate is given by
\[ \frac{Q_i - Q}{Q} = 1.26AV - AV = 0.26 \text{ or } 26\% \]

2.
(a) For dynamic similarity, Froude number should be same for model and prototype.

\[ Fr_m = Fr_p \]

or,

\[ \frac{V_m}{\sqrt{g_i l_m}} = \frac{V_p}{\sqrt{g_p l_p}} \]

or

\[ \frac{V_p}{V_m} = \sqrt{\frac{l_p}{l_m}} = \sqrt{30} \]

From the dynamic similarity for wave making resistance, we have

\[ \frac{F_{wp}}{\rho_p V_p^2 l_p^2} = \frac{F_{wm}}{\rho_m V_m^2 l_m^2} \]

or

\[ F_{wp} = F_{wm} \frac{\rho_p}{\rho_m} \left( \frac{V_p}{V_m} \right)^2 \left( \frac{l_p}{l_m} \right)^2 \]

\[ = 5 \times 1 \times 30 \times 900 = 135000 \text{ N} = 135 \text{ kN} \]

Neglecting the viscous resistance due to water in comparison to the wave making resistance, the problem is described by 5 variables as

\[ f(\rho, V, l, g, F) = 0 \]

These variables are expressed by 3 fundamental dimensions M, L, and T. Therefore, the number of \( \pi \) terms \( = 5 - 3 = 2 \)

Using \( D, \rho \) and \( N \) as repeating variables, \( \pi \) terms can be written as

\[ \pi_1 = l^a \rho^b V^c F \] (2)

\[ \pi_2 = l^a \rho^b V^c g \] (3)

Substituting the variables of Eqs (2-3) in terms of their fundamental dimensions M, L, and T, we get

\[ M^a L^b T^c = \left[ L \right]^a \left[ ML^{-3} \right]^b \left[ LT^{-1} \right]^c \left[ MLT^{-2} \right] \] (4)

\[ M^a L^b T^c = \left[ L \right]^a \left[ ML^{-3} \right]^b \left[ LT^{-1} \right]^c \left[ LT^{-2} \right] \] (5)

Equating the exponents of M, L, and T from Eq.(4), we have

\[ b_i + 1 = 0 \]

\[ -c_i - 2 = 0 \]

\[ a_i - 3b_i + c_i + 1 = 0 \]

which give

\[ a_i = -2, \ b_i = -1, \ c_i = -2 \]

Substituting these values into Eq. (2), we have

\[ \pi_i = \frac{F}{\rho V^2 l^2} \]
Equating the exponents of M, L and T from Eq. (5), we have

\[ b_2 = 0 \]
\[ -c_2 - 2 = 0 \]
\[ a_2 - 3b_2 + c_2 + 1 = 0 \]

which give \( a_2 = 1, \quad b_2 = 0, \quad c_2 = -2 \)

Substituting these values into Eq. (3), we have

\[ \pi_2 = \frac{gl}{V^2} \]

Hence the problem of ship resistance where the viscous resistance due to water is neglected in comparison to the wave making resistance, can be expressed as

\[ f \left( \frac{F}{\rho V^2 l^2}, \frac{gl}{V^2} \right) = 0 \]

or

\[ F = \rho V^2 l^2 \phi \left( \frac{gl}{V^2} \right) \]

(b)

Applying energy equation between sections 1 and 2, we obtain

\[ \frac{p_{am}}{\rho g} + \frac{V_1^2}{2g} + h(t) = \frac{p_{am}}{\rho g} + \frac{\bar{V}(t)^2}{2g} + 0 + h_i \]

(i) Neglecting \( \frac{\bar{V}(t)^2}{2g} \) and entry loss (given) and as \( V_i \) is negligible as compared to \( \bar{V}(t) \), the above equation becomes

\[ h(t) = h_i \]

Considering laminar flow, we have

\[ h_i = \frac{32\mu \bar{V}L}{\rho g D^2} \]

Thus, one can write

\[ h(t) = \frac{32\mu \bar{V}L}{\rho g D^2} \]

or

\[ \bar{V} = \frac{\rho g D^2}{32\mu \bar{V}L} h(t) \]

Again, from continuity equation, we get

\[ A_{pipe} \bar{V} = -A_0 \frac{dh(t)}{dt} \]

or

\[ \frac{\pi D^2 \rho g D^2}{32\mu L} h(t) = -A_0 \frac{dh(t)}{dt} \]

Integrating the above equation, we obtain

\[ h(t) = h_0 \exp \left( \frac{-\pi D^4 \rho g t}{128\mu L A_0} \right) \]
(ii) Given that the flow is highly turbulent, therefore friction factor \( f \approx \) constant.

From energy equation with the consideration of exit kinetic energy, we have

\[
h(t) = \frac{V(t)^2}{2g} + f \frac{L}{D} \frac{V(t)^2}{2g}
\]

or

\[
V(t) = \frac{2gb(t)}{\sqrt{1 + f \frac{L}{D}}}
\]

Again, from continuity equation, we get

\[
A_{pipe} V = -A_0 \frac{dh(t)}{dt}
\]

or

\[
\frac{\pi}{4} D^2 \sqrt{2gb(t)} = -A_0 \frac{dh(t)}{dt}
\]

Integrating the above equation, we obtain

\[
h(t) = \sqrt{h_0} - \frac{\pi D^2 t}{8 A_0} \sqrt{\frac{2g}{1 + f \frac{L}{D}}}
\]

(c)

(i) The isotropic part of the stress tensor is not related to the deformation and it is orientation invariant, while the anisotropic part of the stress tensor is related to the deformation,

(ii) Principal stresses are \( a/2 \) and \(-3a/2\) and all directions are principal directions. State of stress is purely normal and no shear. Pressure is \(-a/2\).

Anisotropic components represent here viscous normal stresses.

3.

(a)

The flow arrangement is shown in the figure below.
Large lateral width of the plate renders the basic flow consideration to be two dimensional, for which the continuity equation under incompressible flow conditions reads:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

For fully developed flow, $$\frac{\partial u}{\partial x} = 0$$

Hence, $$\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow v = v(y)$$

Since $$v = 0$$ at $$y = \pm H$$ as a result of the no-penetration at the walls, $$v$$ is identically equal to zero for all $$y$$, i.e.,

$$\therefore v = 0$$ at $$-H \leq y \leq H$$

Now, considering x-momentum equation, we have, for steady flow,

$$\rho \left[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = -\frac{dp}{dx} + \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Using derivations identical to those presented in Section 8.3.1, it follows

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu} \frac{dp}{dx} = \text{constant}$$

Function of $$y$$ only

Function of $$x$$ only

For $$-H \leq y \leq 0$$

$$\frac{d^2 u_1}{dy^2} = \frac{1}{\mu_1} \frac{dp}{dx}$$

Integrating the above equation, we have

$$\frac{du_1}{dy} = \frac{1}{\mu_1} \frac{dp}{dx} y + C_1$$

$$u_1 = \frac{1}{\mu_1} \frac{dp}{dx} \frac{y^2}{2} + C_1y + C_2$$

(6)

where $$C_1$$ and $$C_2$$ are constants of integration.

For $$0 \leq y \leq H$$

$$\frac{d^2 u_2}{dy^2} = \frac{1}{\mu_2} \frac{dp}{dx}$$

Integrating the above equation, we have

$$\frac{du_2}{dy} = \frac{1}{\mu_2} \frac{dp}{dx} y + C_3$$

$$u_2 = \frac{1}{\mu_2} \frac{dp}{dx} \frac{y^2}{2} + C_3y + C_4$$

(7)

where $$C_3$$ and $$C_4$$ are constants of integration.

Eqs (6) and (7) are subjected to the following boundary conditions
At \( y = -H \), \( u_1 = 0 \),

At \( y = H \), \( u_2 = 0 \),

At \( y = 0 \), \( u_1 = u_2 \) (continuity of flow velocity)

At \( y = 0 \), \( \mu_1 \frac{du_1}{dy} = \mu_2 \frac{du_2}{dy} \) (continuity of shear stress)

From the boundary conditions at \( y = -H \), \( u_1 = 0 \), and at \( y = H \), \( u_2 = 0 \), we get

\[
C_2 = C_1 H - \frac{1}{\mu_1} \frac{dp}{dx} \frac{H^2}{2}
\]

\[
C_4 = -C_1 H - \frac{1}{\mu_2} \frac{dp}{dx} \frac{H^2}{2}
\]

From the boundary condition at \( y = 0 \), \( \mu_1 \frac{du_1}{dy} = \mu_2 \frac{du_2}{dy} \), we have

\[
\mu_1 C_1 = \mu_2 C_3
\]

\[
\Rightarrow C_1 = \frac{\mu_2}{\mu_1} C_3
\]

From the boundary condition at \( y = 0 \), \( u_1 = u_2 \), one can write

\[
C_2 = C_4
\]

or,

\[
C_1 H - \frac{1}{\mu_1} \frac{dp}{dx} \frac{H^2}{2} = -C_1 H - \frac{1}{\mu_2} \frac{dp}{dx} \frac{H^2}{2}
\]

or,

\[
(C_1 + C_1) H = \frac{dp}{dx} \frac{H^2}{2} \left[ \frac{1}{\mu_1} - \frac{1}{\mu_2} \right]
\]

or,

\[
\left( \frac{\mu_2 + 1}{\mu_1} \right) C_3 = \frac{dp}{dx} \frac{H}{2} \left[ \frac{1}{\mu_1} - \frac{1}{\mu_2} \right]
\]

or,

\[
C_3 = \frac{dp}{dx} \frac{H (\mu_2 - \mu_1)}{2 (\mu_2 + \mu_1) \mu_2}
\]

or,

\[
C_1 = \frac{dp}{dx} \frac{H (\mu_2 - \mu_1)}{2 (\mu_2 + \mu_1) \mu_1}
\]

Substituting the values of \( C_1 \), \( C_2 \), \( C_3 \) and \( C_4 \) in the Eqs (6) and (7), we have

\[
u_1 = \frac{1}{\mu_1} \frac{dp}{dx} \frac{1}{2} \left[ y^2 - H^2 + \frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)} H (y + H) \right]
\]

\[
u_2 = \frac{1}{\mu_2} \frac{dp}{dx} \frac{1}{2} \left[ y^2 - H^2 + \frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)} H (y - H) \right]
\]

Qualitative sketch of the velocity profiles are shown in the figure below.
Note from the above figure that since \( \mu_x < \mu_1 \), \( \frac{du_1}{dy} \bigg|_{y=0} > \frac{du_2}{dy} \bigg|_{y=0} \) (so as to satisfy

\[
\mu_1 \frac{du_1}{dy} \bigg|_{y=0} = \mu_2 \frac{du_2}{dy} \bigg|_{y=0}
\]

or

\[
\frac{dy}{du_1} \bigg|_{y=0} > \frac{dy}{du_2} \bigg|_{y=0},
\]

or equivalently \( \tan \theta_1 > \tan \theta_2 \).

(b)
It is given that

\[
p = p_x + \frac{1}{2} \rho U_x^2 (1 - 4 \sin^2 \theta)
\]

for \( 0 \leq \theta \leq 120^\circ \)

At \( \theta = 120^\circ \)

\[
p = p_x + \frac{1}{2} \rho U_x^2 (1 - 4 \sin^2 120^\circ) = p_x - \rho U_x^2
\]

It is given that the pressure in the wake region remains approximately constant that means

\[
p = p_x - \rho U_x^2
\]

for \( 120^\circ \leq \theta \leq 180^\circ \)

Consider an elemental area subtending an angle \( d\theta \) as shown in the figure below. Drag force acting on the elemental area due to pressure (neglecting the skin frictional drag) is

\[
dF_D = pdA \cos \theta = p \cos \theta Rd\theta L
\]

(8)
The total drag force acting on the cylinder is obtained by integrating Eq. (8) over the entire surface of the cylinder as

\[
F_D = \int dF_D = 2 \int_{\theta=0}^{\theta=120^\circ} p \cos \theta R L d \theta + 2 \int_{\theta=0}^{\theta=180^\circ} p \cos \theta R L d \theta
\]

\[
= 2 \int_{\theta=0}^{\theta=120^\circ} \left( p_\infty + \frac{1}{2} \rho U^2_\infty \left( 1 - 4 \sin^2 \theta \right) \right) \cos \theta R L d \theta + 2 \int_{\theta=0}^{\theta=180^\circ} \left( p_\infty - \rho U^2_\infty \right) \cos \theta R L d \theta
\]

\[
= 2RL \left( p_\infty + \frac{1}{2} \rho U^2_\infty \right) \int_{\theta=0}^{\theta=120^\circ} \cos \theta d \theta - 4 \rho U^2_\infty RL \int_{\theta=0}^{\theta=120^\circ} \sin^2 \theta \cos \theta d \theta
\]

\[
+ 2 \left( p_\infty - \rho U^2_\infty \right) RL \int_{\theta=0}^{\theta=180^\circ} \cos \theta d \theta
\]

\[
= 2RL \left( p_\infty + \frac{1}{2} \rho U^2_\infty \right) \sin \theta_{120^\circ} - 4 \rho U^2_\infty RL \frac{\sin^3 \theta_{120^\circ}}{3} + 2 \left( p_\infty - \rho U^2_\infty \right) RL \sin \theta_{120^\circ}
\]

\[
= 2RL \left( p_\infty + \frac{1}{2} \rho U^2_\infty \right) \frac{\sqrt{3}}{2} - 4 \rho U^2_\infty RL \frac{3\sqrt{3}}{8} - 2 \left( p_\infty - \rho U^2_\infty \right) RL \frac{\sqrt{3}}{2}
\]

\[
= 2RL\rho U^2_\infty \frac{\sqrt{3}}{2} = \sqrt{3} RL\rho U^2_\infty
\]

The drag coefficient is found to be

\[
C_D = \frac{F_D}{\frac{1}{2} \rho U^2_\infty A} = \frac{\sqrt{3} RL\rho U^2_\infty}{\frac{1}{2} \rho U^2_\infty 2RL} = \sqrt{3}
\]

4.

(a)

Free stream velocity is given as \( U_\infty = cx^m \)

Outside the boundary layer, flow is irrotational, and hence the Bernoulli’s equation is valid. Bernoulli’s equation can be written as

\[
p_\infty + \frac{1}{2} \rho U^2_\infty = \text{constant}
\]

Differentiating the above equation with respect to \( x \), we get

\[
\frac{dp_\infty}{dx} + \rho U_\infty \frac{dU_\infty}{dx} = 0
\]

or,

\[
\frac{1}{\rho} \frac{dp_\infty}{dx} = -U_\infty \frac{dU_\infty}{dx} = -cx^m \left( cmx^{m-1} \right) = -c^2 mx^{2m-1}
\]

Boundary layer equations for a constant density, two-dimensional, steady flow are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp_\infty}{dx} + \frac{\partial^2 u}{\partial y^2}
\]
Substituting the value of $\frac{1}{\rho} \frac{dp}{dx}$ in momentum equation, we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2}$$

(b)
The variation of wall shear stress as a function of axial position along the plate is shown in the figure below.

![Wall shear stress, $\tau_w$](image)

(c)
The velocity profile is given as

$$\frac{u}{U_{\infty}} = a_0 + a_1 \frac{y}{\delta}$$

The velocity profile must satisfy the essential boundary conditions i.e.,
at $y = 0$, $u = 0$ (no-slip condition at the plate)
at $y = \delta$, $u = U_{\infty}$ (free stream velocity at the edge of the boundary layer)

Applying the above two boundary conditions, one gets

$a_0 = 0$ and $a_1 = 1$

The velocity profile is given by

$$\frac{u}{U_{\infty}} = \frac{y}{\delta}$$

The boundary layer equations for steady, two-dimensional, constant density, laminar flow over a flat plate flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(9)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

(10)

Integrating Eq.(9) with respect to $y$ within boundary layer (from $y = 0$ to $y = \delta$), we get

$$\frac{\partial u}{\partial x} \bigg|_0^\delta + \frac{\partial u}{\partial y} \bigg|_0^\delta = 0$$
or 
\[ \int_0^\delta \frac{\partial u}{\partial x} dy + [v]^\delta_0 = 0 \]

or 
\[ \int_0^\delta \frac{\partial u}{\partial x} dy + v\bigg|_{y=\delta} - v|_{y=0} = 0 \]

or 
\[ \int_0^\delta \frac{\partial u}{\partial x} dy + v\bigg|_{y=\delta} = 0 \]

\[ \therefore v|_{y=0} = 0 \text{ because of no-penetration boundary condition} \]

or 
\[ v|_{y=\delta} = -\int_0^\delta \frac{\partial u}{\partial x} dy \] (11)

Integrating Eq.(10) with respect to y within boundary layer (from \( y = 0 \) to \( y = \delta \), we get

\[ \int_0^\delta \frac{\partial u}{\partial x} dy + \int_0^\delta \frac{\partial u}{\partial y} dy = \int_0^\delta v \frac{\partial u}{\partial x} dy \]

or 
\[ \int_0^\delta \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) dy = \int_0^\delta v \frac{\partial u}{\partial x} dy \]

or 
\[ \int_0^\delta \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) dy = \int_0^\delta v \frac{\partial u}{\partial y} dy \]

or 
\[ \int_0^\delta \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) dy = -v \frac{\partial u}{\partial y} \bigg|_{y=0} \]

or 
\[ \int_0^\delta (u^2 - U_\infty u) dy = -\frac{\mu}{\rho} \frac{\partial u}{\partial y} \bigg|_{y=0} \]

Applying Leibnitz rule, the above equation becomes

\[ \frac{d}{dx} \int_0^\delta \left( u^2 - U_\infty u \right) dy = -\frac{\mu}{\rho} \frac{\partial u}{\partial y} \bigg|_{y=0} \]

or 
\[ \frac{d}{dx} \int_0^\delta u \left( 1 - \frac{u}{U_\infty} \right) dy = -\frac{\mu}{\rho U_\infty^2} \frac{\partial u}{\partial y} \bigg|_{y=0} \]

Substituting \( \frac{u}{U_\infty} = \frac{y}{\delta} \), we obtain

\[ \frac{d}{dx} \int_0^\delta \left( 1 - \frac{y}{\delta} \right) dy = -\frac{\mu}{\rho U_\infty^2} \frac{U_\infty}{\delta} \]
or \[ \frac{1}{6} \frac{d\delta}{dx} = \frac{\mu}{\rho U_\infty \delta} \]
or \[ \delta d\delta = \frac{6\mu}{\rho U_\infty} dx \]

Integrating the above equation, we get
\[ \frac{\delta^2}{2} = \frac{6\mu}{\rho U_\infty} x + C \]
where \( C \) is an integration constant.
At \( x \to 0, \delta \to 0 \), so \( C = 0 \).

\[ \frac{\delta^2}{2} = \frac{12\mu}{\rho U_\infty} x \]
or
\[ \delta = \sqrt{\frac{12\mu}{\rho U_\infty}} x \]

The total shear force on one side of the plate is given by
\[ F_D = \int_0^L \tau_w w dx \]
\[ = \int_0^L \frac{\mu}{\delta} U_x dy \]
\[ = \frac{\rho U_\infty^2 w}{2} \sqrt{\frac{\mu}{3\rho U_\infty}} \int_0^L x \frac{1}{3} dx \]
\[ = \rho U_\infty^2 w \sqrt{\frac{\mu}{3\rho U_\infty}} L^2 \]

Drag coefficient is then
\[ C_D = \frac{F_D}{\frac{1}{2} \rho U_\infty^2 w L} = \frac{\rho U_\infty^2 b}{\frac{1}{2} \rho U_\infty^2 w L} \]
\[ = \frac{2}{\sqrt{3}} \sqrt{\frac{\mu}{\rho U_\infty L}} = \frac{1.15}{\sqrt{\text{Re}_L}} \]

(d)
The boundary layer equations for steady, two-dimensional, constant density, laminar flow over a flat plate flow are
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (12) \]
\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (13) \]
From the scaling estimation of Eq.(12) one can write
\[ \frac{U_\infty}{L} \frac{\nu}{\delta} \]
From the scaling estimation of Eq.(13) one can write

\[ -U_\infty \frac{U_\infty}{L} - v_b \frac{U_\infty}{\delta} - \nu \frac{U_\infty}{\delta^2} \]

or

\[ \frac{U_\infty^2}{L} - \nu \frac{U_\infty}{\delta^2} \]

or

\[ \delta \sim \left( \frac{\nu L}{U_\infty} \right)^{1/2} \]

From the scaling estimation, one can write

\[ \tau_w \sim \mu \frac{U_\infty}{\delta} \]

\[ C_{ly} \sim \frac{\tau_w}{\rho U_\infty^2} \]

or

\[ C_{ly} \sim \frac{\mu U_\infty}{\rho \delta^2} \sim \frac{1}{\rho U_\infty} \frac{\mu}{\rho U_\infty^2} \sim \frac{1}{\rho U_\infty} \left( \frac{\nu L}{U_\infty} \right)^{1/2} \]

or

\[ \frac{C_{ly,1}}{C_{ly,2}} = \left( \frac{L_2}{L_1} \right)^{1/2} = \sqrt{\frac{1}{2}} \]

(e)

The variations of \( u, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2} \), as a function of \( y \) (where \( y \) is the boundary layer coordinate) for flow over a surface with favourable pressure gradient are shown in the figure below.