Full State Feedback Control (non-canonical)

Dr. Bishakh Bhattacharya
Professor, Department of Mechanical Engineering
IIT Kanpur
This Lecture Contains

- Full State Feedback Control for System in Non-canonical form
- Ackermann’s algorithm
- Bass-Gura formulation
- Where to place the closed loop poles?
- Assignment
Full state feedback control for system in non-canonical form

• If the system is not in control canonical form, you have to find out the proper transformation matrix $T$ to convert the system into canonical form.

• If $x$ is the state vector corresponding to non-canonical form along with the corresponding state-space parameters $A$, $B$ and $C$ and $z$ is the state vector in canonical form along with system parameters given by $A_c$, $B_c$ and $C_c$, then, considering $T$ to be the transformation matrix between the two linear systems such that:

$$x = Tz \quad \text{then the state space equation in non – canonical form}$$

$$\dot{x} = Ax + Bu, \text{ gets transformed to Canonical form as}$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu = A_c z + B_c u$$
Full state feedback control in non-canonical form contd..

• The first task here is to find out the controllability matrix corresponding to the canonical form.

• How do we find it without knowing the transformation matrix?

• Well, we can find out the roots of the characteristic equation by evaluating the determinant of \([sI-A]^{-1}\)

• Once we know the roots, we can write the new plant matrix in canonical form (see the standard form discussed before)

• In order to obtain the controllability matrix you also need to know the \(B\) matrix, for a single input system it is simply

\[
B = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T
\]
A System not in Control Canonical Form

After evaluating the controllability matrix related to the canonical form, you can find the controllability matrix corresponding to the non-canonical form as

\[ \hat{C} = \begin{bmatrix} B & A & B & \cdots & A^{n-1}B \end{bmatrix} = T\hat{C}_c \]

This controllability matrix can be used along with the controllability matrix corresponding to canonical form to obtain the transformation matrix between the two systems as:

\[ T = \hat{C} \hat{C}_c^{-1} \]

Now, you can represent the system to canonical form and obtain the corresponding gain as \( K_c \). Then, the gain for non-canonical form \( K \) could be written as

\[ K = K_c T^{-1} \]
Controller Design using Ackermann’s algorithm

For a single input system, one can use a direct relationship to find the controller gain $K$ by using Ackermann’s formulation as follows:

$$K = R \hat{C}^{-1} \Psi(A)$$

with $R = \begin{bmatrix} 0 & \cdots & \cdots & 1 \end{bmatrix}$,

$\hat{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$

and $\Psi(A) = A^n + d_{n-1}A^{n-1} + d_{n-2}A^{n-2} + \cdots + d_0I$

where, $d_i$ are the coefficients of the desired characteristic polynomial.

This is based on the fact that a matrix satisfies it’s own characteristic equation, which is also known as Cayley-Hamilton’s theorem.
Controller Design using Bass-Gura algorithm

For a single input system, when the characteristic polynomial of the open-loop plant \( [a_0 \quad a_1 \quad \ldots \quad a_n] \) and the desired characteristic Polynomial \( [d_0 \quad d_1 \quad \ldots \quad d_n] \) are known beforehand, you can use the well-known Bass-Gura technique to obtain the control gain vector \( \mathbf{K} \) as

\[
\mathbf{K} = [(\hat{\mathbf{C}} \mathbf{W})^T]^{-1} (\Psi - \hat{\Psi})
\]

\[
\mathbf{W} = \begin{bmatrix}
1 & a_0 & \cdots & a_{n-1} \\
0 & 1 & a_{n-2} \\
\vdots & \vdots & \vdots & \ddots \\
0 & \cdots & \cdots & 1
\end{bmatrix}
\]

\[
\Psi = [d_0 \ldots d_{n-1}], \quad \hat{\Psi} = [a_0 \ldots a_{n-1}]
\]
Where to place the Closed-loop poles?

- The placement of the pole often becomes one of the important prerogatives of the controller design. Given a freedom, you should design a system such that it is predominantly second order in nature. This implies that the higher order poles should be placed at least five times away from the real part of the second order poles.
- However, from the energy point of view, you should not place the closed-loop poles quite far away from the open loop poles as the gain requirement would increase proportionately.
- The choice of B matrix also places an important role as the lesser controllable systems require higher gains.
Butterworth pole configurations

Following an optimization procedure, it is shown that the closed loop poles could be placed such that the characteristic equation is

\[
\left( \frac{s}{\omega} \right)^{2k} = (-1)^{k+1}
\]

Where, \( k \) is the number of poles required.

It can be shown that for \( k=1 \), you need to place a single pole on the –ve real axis at a distance \( \omega \) from the origin. For, \( k=2 \), the radial distance remains unchanged, however, the poles will be complex and at angle 45\(^0\) from the imaginary axis. These Configurations are known as Butterworth pole configuration.
Assignment:

A SDOF system has the following mass, stiffness and damping constant in appropriate units, $m=0.1$, $c=0.01$ and $k=0.8$; design a full-state feedback control, with an actuator influence matrix $B^T = [0 \ 1]$ and a forcing function $0.1u(t)$ ($u(t)$ – unit step function), such that the desired eigen-values are at $-1 \pm 2j$, respectively.
Special References for this lecture

- Control System Design, Bernard Friedland, Dover

- Control Systems Engineering – Norman S Nise, John Wiley & Sons

- Design of Feedback Control Systems – Stefani, Shahian, Savant, Hostetter

  Oxford