Module 2
Selection of Materials and Shapes
Lecture 6
Co-Selection of Materials and Shapes
Instructional objectives

By the end of this lecture, the student will learn
(a) how to modify the material indices obtained in lecture 2 to include shape factors,
(b) how to make a trade-off between material properties and shape factors,

Material indices that include Shape Factor

So far we have obtained material indices, also referred to as the performance indices, and shape factors independently for different structural applications. We have observed that the material indices are not influenced by the shape. However, the resistance to stiffness and strength are significantly influenced by the shape factors. Thus, the overall aim of the selection of a suitable material for a given application would be facilitated if the material indices can be compounded with appropriate shape factors. We will show in the following how combined material indices can be formed considering both the important properties of material and attributes of shape.

Example 12: Combined Material Index for elastic bending

Consider the selection of a material for a beam of specified bending stiffness $S_B$ and length $L$ (the constraints), to have minimum mass, $m$ (the objective). The mass $m$ of a beam of length $L$ and section area $A$ is given, as before, by

$$m = AL\rho \quad (1)$$

Before going to develop combined material index, the methodology has to be clarified. From the elemental beam theory, we get the expression for bending stiffness. The bending stiffness, $S_B$, is given by

$$S_B = \frac{C_1EI}{L^3} \quad (2)$$

where $C_1$ is a constant that depends only on the way the loads are distributed on the beam. Now, in this case the bending stiffness is specified and we have to incorporate the shape factor and material properties in the combined index. So, replacing $I$ from $\phi_B^e = \frac{4\pi I}{A^2}$ in equation (2), we can write
$$S_B = \frac{C_1 E \phi_B^e A^2}{4\pi L^3} \quad (3)$$

From the problem statement it is clear that the cross section of the beam is also a variable. So, $A$ has to be replaced from equation (1) to obtain the expression of $m$ in proper form. Using equations (1) and (3) and replacing $A$ from equation (1),

$$m = \left( \frac{4\pi S_B}{C_1} \right)^{1/2} \cdot L^{5/2} \cdot \left( \frac{\rho^2}{E \phi_B^e} \right)^{1/2} \quad (4)$$

So, to minimize the mass, we have to maximize the combined material index, $M_1$

$$M_1 = \frac{(E \phi_B^e)^{1/2}}{\rho} \quad (5)$$

Example 13: Combined Material Index for elastic torsion

The procedure for the development of combined material index for elastic twisting of shafts is similar. Let us consider a shaft of cross-sectional area $A$ and length $L$ which is subjected to a torque $T$ and twists through an angle, $\theta$. It is required that the torsional stiffness, $T/\theta$, meets a specified target, $S_T$, at the minimum mass. The methodology is exactly same as that of previous example. Only the bending case has been replaced by torsion and the corresponding changes has been done. The torsional stiffness, $S_T$, can be expressed as

$$S_T = \frac{KG}{L} \quad (6)$$

where $G$ is the shear modulus and $K$ is related to the shape factor under elastic torsion that is given as $\phi_T^e = \left( \frac{2\pi K}{A^2} \right)$. Equation (6) can therefore be rearranged as

$$S_T = \frac{G \phi_T^e A^2}{2\pi L} \quad (7)$$

Next, using equation (7) to eliminate $A$ in equation (1) leads to

$$m = \left( 2\pi S_T \right)^{1/2} \cdot L^{3/2} \cdot \left( \frac{\rho^2}{G \phi_T^e} \right)^{1/2} \quad (8)$$
By using $G \approx \frac{3}{8}E$, we note that the combined material index ($M_2$) that needs to be maximized to minimize the mass can be written as:

$$M_2 = \frac{(E \phi_T)^{1/2}}{\rho} \quad (9)$$

**Example 14: Combined Material Index for design against failure in bending**

The problem statement remains almost similar to the previous two examples. A beam of length $L$, loaded in bending, must support a specified load $F$ without failure and be as light as possible. When the section-shape is also a variable along with the material, a combined material index can be found as follows. Failure occurs if the load exceeds the bending moment ($M_f$)

$$M_f = Z \sigma_f \quad (10)$$

where $Z$ is the section modulus and $\sigma_f$ is the failure strength of any material. Replacing $Z$ by the shape factor $\phi_B$ in equation (10), we get

$$M_f = \frac{\sigma_f \phi_B A^{3/2}}{4\sqrt{\pi}} \quad (11)$$

Substituting this into equation (1) for the mass of the beam gives

$$m = (4\sqrt{\pi} M_f)^{2/3} L \cdot \left( \frac{\rho^{3/2}}{\sigma_f \phi_B} \right)^{2/3} \quad (12)$$

Hence, the suitable combined material index ($M_3$) that needs to be maximized in this case to identify a suitable material considering its best possible shape factor can be given as

$$M_3 = \frac{(\sigma_f \phi_B)^{2/3}}{\rho} \quad (13)$$

**Example 15: Combined Material Index for design against failure in torsion**

Similar to bending, the failure of the beam will occur if the load exceeds the torque

$$T_f = \tau_f Q \quad (14)$$

Replacing $Q$ by the shape factor $\phi_T$ in equation (14), we get
\[
T_f = \frac{\sigma_f \phi_f^f A^{3/2}}{4\sqrt{\pi}} \tag{15}
\]

Substituting this into equation (1) for the mass of the beam yields
\[
m = \left(\frac{4\sqrt{\pi}T_f}{2} \right)^{2/3} \cdot L \cdot \left(\frac{\rho^{3/2}}{\sigma_f \phi_f^f} \right)^{2/3} \tag{16}
\]

The best material-and-shape combination would confirm to the greatest value of the material index \((M_4)\)
\[
M_4 = \frac{(\sigma_f \phi_f^f)^{2/3}}{\rho} \tag{17}
\]

**Example 16: Selection of Material considering shape factor for Elastic Bending**

Let us consider an example of the selection of a material for a stiff, shaped beam of minimum mass. *Table 1* enlists four materials that are available. We will search for largest value of the combined material index, \(M_1\), as noted in equation (5). It is imperative from *Table 1* that wood would be chosen as the best material if the shape factor is not considered. However, considering both material properties and the shape factor, wood becomes the worst choice and AA 6061-T4 becomes the most suitable material. This happens primarily because of the fact that the maximum value of shape factor that can be obtained in wood is limited by manufacturability of wood material.

<table>
<thead>
<tr>
<th>Material</th>
<th>(\rho) (Mg/m(^3))</th>
<th>(E) (GPa)</th>
<th>(\phi_B^e)</th>
<th>(\frac{E^{1/2}}{\rho})</th>
<th>(\frac{(E\phi_B^e)^{1/2}}{\rho})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1020 Steel</td>
<td>7.85</td>
<td>205</td>
<td>20</td>
<td>1.8</td>
<td>8.2</td>
</tr>
<tr>
<td>AA 6061- T4</td>
<td>2.7</td>
<td>70</td>
<td>15</td>
<td>3.1</td>
<td>12.0</td>
</tr>
<tr>
<td>GFRP</td>
<td>1.75</td>
<td>28</td>
<td>8</td>
<td>2.9</td>
<td>8.5</td>
</tr>
<tr>
<td>Wood</td>
<td>0.9</td>
<td>13.5</td>
<td>2</td>
<td>4.1</td>
<td>5.8</td>
</tr>
</tbody>
</table>

The combined material index for elastic bending can be rewritten as
\[ M_1 = \frac{(\phi_B^e E)^{1/2}}{\rho} = \frac{(E/\phi_B^e)^{1/2}}{\rho/\phi_B^e} = \frac{(E^*)^{1/2}}{\rho^*} \]  

(18)

i.e. a material with modulus \((E)\) and density \((\rho)\), when given a particular shape can be thought of as a new material with modulus \((E^*)\) and density \((\rho^*)\) presuming \(E^* = \frac{E}{\phi_B^e}\) and \(\rho^* = \frac{\rho}{\phi_B^e}\).

Figure 2.6.1 shows the elastic modulus vs. density \((E \text{ vs. } \rho)\) chart with the modified material properties \(E^*\) and \(\rho^*\) plotted into it. Now, the structured material behaves like a new material with the modified material properties. When the shape factor has not been introduced, the shape factor takes the position \(\phi = 1\). It can be observed that introduction of a shape factor \(\phi_B^e = 10\) moves the combined material index, \(M_1\) to the lower left along a line of slope 1, from the position \((E, \rho)\) to the position \((E/10, \rho/10)\).

![Figure 2.6.1](image)

**Figure 2.6.1** Young’s Modulus vs. density chart with the influence of shape factor \((\phi_B^e)\) for elastic bending [2]
Example 17: Selection of Material considering shape factor for *Failure in Bending*

Similar to the previous example, material index for elastic bending can be rewritten as

\[
M_3 = \frac{\left(\phi_B \sigma_f\right)^{2/3}}{\rho} = \frac{\left(\sigma_f / (\phi_B)^2\right)^{2/3}}{\rho / (\phi_B)^2} = \frac{(\sigma_f^*)^{2/3}}{\rho^*}
\]  \hspace{1cm} (19)

i.e. a material with failure strength, \(\sigma_f\) and density, \(\rho\), when given a particular shape, can be envisaged as a new material with modified strength and density as \(\left(\frac{\sigma_f}{(\phi_B)^2}\right)\) and \(\left(\frac{\rho}{(\phi_B)^2}\right)\), respectively. Figure 2.6.2 shows a typical failure strength \((\sigma_f)\) vs. density \((\rho)\) chart with the modified material properties \((\sigma_f^*\) and \(\rho^*)\) indicated into it. It can be seen in Figure 2.6.2 that the introduction of a typical shape factor \(\phi_B = \sqrt{10}\), moves the combined material index, \(M_2\) to the lower left along a line of slope 1, from the position \((\sigma_f, \rho)\) to the position \((\sigma_f / \sqrt{10}, \rho / \sqrt{10})\)

**Figure 2.6.2**  Strength vs. density chart with the influence of shape factor \((\phi_B^f)\) for failure in bending [2]
Exercise

(1) As done in case of bending, explain the procedure for obtaining modified material properties in case of torsional loading.

(2) Modify all the material indices if the objective is to minimize the cost rather than the mass [Hint: Assume $C_m$ to be cost of material per unit volume.]

References