ME-662 CONVECTIVE HEAT AND MASS TRANSFER

A. W. Date
Mechanical Engineering Department
Indian Institute of Technology, Bombay
Mumbai - 400076
India

LECTURE-27 TURBULENCE MODELS-2

LECTURE-27 TURBULENCE MODELS-2

1. Low Re_t Two-Eqn model
2. High Re_t Stress-Eqn model
3. Low Re_t Stress-Eqn model
4. Algebraic Stress-Eqn model
5. Scalar Transport model
    1. Eddy Diffusivity model
    2. Turbulent Flux Model
6. Modeling Combustion and Turbulence Interaction

Low Re_t e-ϵ model L27($\frac{1}{19}$)

For low $Re_t = \nu_t/\nu$

$$
\begin{aligned}
\rho \frac{De}{\partial t} &= \frac{\partial}{\partial x_i}\left\{(\mu + \frac{\mu_t}{\sigma_e})\frac{\partial e}{\partial x_i}\right\} + \mu_t \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \frac{\partial u_i}{\partial x_j} - \rho\, \epsilon^* \\
\rho \frac{D\epsilon^*}{\partial t} &= \frac{\partial}{\partial x_i}\left\{(\mu + \frac{\mu_t}{\sigma_\epsilon})\frac{\partial \epsilon^*}{\partial x_i}\right\} \\
&- \frac{\epsilon^*}{e}\left\{C_1\, \mu_t \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \frac{\partial u_i}{\partial x_j} - C_2\, \rho\, \epsilon^*\right\} \\
&+ 2\, \nu\, \mu_t \, (\frac{\partial^2 u_i}{\partial x_k\, \partial x_l})^2 \\
\mu_t &= C_D^* \, (\frac{\rho\, e^2}{\epsilon^*}), \quad C_D^* = C_D \, \exp\left\{\frac{-3.4}{(1 + Re_t/50)^2}\right\} \\
C_1^* &= C_1, \quad C_2^* = C_2 \, \left[1 - 0.3\, \exp\left\{-\, Re_t^2\right\}\right] \\
\epsilon^* &= \epsilon - 2\, \nu\, (\frac{\partial e^{0.5}}{\partial x_i})^2
\end{aligned}
$$

Comments - L27($\frac{2}{19}$)

1. Model constants[1] are sensitised to low Re_t region near the wall. They tend to high Re_t values beyond sub-layers
2. The correction to C_D is chosen to give values of ν_t in agreement with the Van-Driest mixing length formula
3. The correction to C_2 is selected from exptl. data on the decay of isotropic turbulence at low Re_t (at large times, $e \propto t^{-n}$ where n $\simeq$ 2.5 to 2.8).
4. The correction to ϵ is introduced to account for the non-isotropic contribution to the dissipation.
5. Wall-functions are no longer necessary and e and ϵ Eqns can be solved with $e_{wall} = \epsilon^*_{wall} = 0$. However, to capture the low Re_t effects, very fine mesh ($>$ 60 grid nodes) become necessary in the $y^+ <$ 100 region.

[1] Jones W P and Launder B L *The Prediction of Laminarisation with a Two-Equation Model of Turbulence*, Int. Jnl. of Heat and Mass Transfer, vol. 15, p 301, 1972

Stress Eqn Model- L27($\frac{3}{19}$)

Six transport equations for the one-point correlation $\overline{u_i' \, u_j'}$ are derived from equation for B_{ij} by setting separation $\xi_k = 0$ (lecture 23)

$$\frac{D\, \overline{u_i' \, u_j'}}{Dt} = - \left[\overline{u_j' \, u_k'} \, \frac{\partial u_i}{\partial x_k} + \overline{u_i' \, u_k'} \, \frac{\partial u_j}{\partial x_k} \right]$$

$$\{P_{ij}\}$$

$$- \frac{\partial}{\partial x_k} \left[\overline{u_i' \, u_j' \, u_k'} + \overline{\frac{p'}{\rho} \left\{ u_i' \, \delta_{jk} + u_j' \, \delta_{ik} \right\}} \right]$$

$$\{D_{ij}\}$$

$$+ \overline{\frac{p'}{\rho} \left\{ \frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right\}} - 2\,\nu \, \overline{\frac{\partial u_i'}{\partial x_k} \, \frac{\partial u_j'}{\partial x_k}}$$

$$\{PS_{ij}\} \qquad \{\epsilon_{ij}\}$$

Modeling $\overline{u_i' u_j'}$ Eqn - L27($\frac{4}{19}$)

1. Invoking the idea of local isotropy at high Re_t the destruction rate is equally distributed among all its components . Hence $\epsilon_{ij} = (2/3)\,\epsilon\,\delta_{ij}$ where ϵ is obtained from its eqn.
2. Pressure-Strain Correlation PS_{ij} acts in two ways: Firstly, it sustains the division of TKE (e) into its three components $\overline{u_i'^2}$ and secondly, it *destructs* the absolute magnitude of the shear stresses. Hence, without further elaboration

$$
\begin{aligned}
-PS_{ij} &= C_{p_1}\,\frac{\epsilon}{e}\,(\overline{u_i'\,v_j'} - \frac{2}{3}\,e\,\delta_{ij}) + C_{p2}\,(P_{ij} - \frac{P_{ii}}{3}) \\
&+ C_{p3}\,(P_{ij}' - \frac{2}{3}\,P\,\delta_{ij}) + C_{p4}\,e\,(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + PS_w \\
PS_w &= \frac{e^{3/2}}{\epsilon\,L_B}\left[C_{p1}'\,\frac{\epsilon}{e}\,(\overline{u_i'\,v_j'} - \frac{2}{3}\,e\,\delta_{ij}) + C_{p2}'\,(P_{ij} - P_{ij}')\right] \\
P_{ij}' &= -\overline{u_i'\,u_k'}\,\frac{\partial U_k}{\cdots} - \overline{u_j'\,u_k'}\,\frac{\partial U_k}{\cdots}
\end{aligned}
$$

(see next slide)

Contd . . . - L27($\frac{5}{19}$)

This algebraic expression for PS_{ij} is derived from its exact Eqn[2]. The term containing C_{p1} is called return-to-isotropy . The PS_w term is called the *wall-reflection* term which accounts for the effects of pressure reflections from the wall. The recommended constants are: $C_{p1} = 1.5,\ C'_{p1} = 0.12,\ C_{p2} = 0.764,\ C'_{p2} = 0.01,\ C_{p3} = 0.109,\ C_{p4} = 0.182,\ L_B =$ wall distance. Finally the Triple Velocity correlation $\overline{u'_i\,u'_j\,u'_k}$ in the Diffusion term D_{ij} is modeled from its exact Eqn and $\overline{(p'/\rho)\ \left\{\partial u'_i/\partial x_j + \partial u'_j/\partial x_i\right\}} \simeq 0$.

$$-\overline{u'_i\,u'_j\,u'_k} = C_s\,\frac{e}{\epsilon}\,\left\{\overline{u'_i\,u'_l}\,\frac{\partial\overline{u'_j\,u'_k}}{\partial x_l} + \overline{u'_j\,u'_l}\,\frac{\partial\overline{u'_k\,u'_i}}{\partial x_l} + \overline{u'_k\,u'_l}\,\frac{\partial\overline{u'_i\,u'_j}}{\partial x_l}\right\}$$

where $C_s \simeq 0.08$ to o.11 (from num expts)

[2]Hanjalic K. and Launder B. E. *A Reynolds Stress Model of Turbulence and its Application to Thin Shear Flows*, JFM.,52(4), p 609-638, 1972

Algebraic Models (ASMs) - L27($\frac{6}{19}$)

1. Implementation of Stress-Eqn model requires solution of 6 differential eqns for $\overline{u_i' \, u_j'}$, 2 Eqns for e and ϵ coupled with the 3 RANS Eqns. This is a formidable problem.
2. The modeled forms presented above show that spatial gradients of $\overline{u_i' \, u_j'}$ occur only in the diffusion and convection - these terms make the Eqns differential ones.
3. Alg. Stress Models are developed using the idea that

$$
\begin{aligned}
\frac{\overline{u_i' \, u_j'}}{e} &\simeq \frac{\frac{D \, \overline{u_i' \, u_j'}}{Dt} - \text{Diff}\,(\overline{u_i' \, u_j'})}{\frac{D\,e}{D\,t} - \text{Diff}\,(e)} = \frac{-(2/3)\,(1 - C_{p1})\delta_{ij} + (P/\,\epsilon)F}{(P/\,\epsilon) - 1 + C_{p1}} \\
F &= (1 - C_{p2})\,\frac{P_{ij}}{P} - C_{p3}\,\frac{P_{ij}'}{P} - C_{p4}\,\frac{e}{P}\,(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \\
&+ \frac{2}{3}\,(C_{p2} + C_{p3})\,\delta_{ij} \quad (\text{ computational expense reduced})
\end{aligned}
$$

Low Re_t ASM - L27($\frac{7}{19}$)

$$
\begin{aligned}
\overline{u_i' \, u_j'} &= -(2/3)\, e \, \delta_{ij} + e \times F \\
F &= \frac{\nu_t}{e} S_{ij} + C_1 \frac{\nu_t}{e} \left(S_{ik} S_{jk} - \frac{1}{3} S_{kl} S_{kl} \delta_{ij}\right) \\
&+ C_2 \frac{\nu_t}{e} \left(\Omega_{ik} S_{jk} + \Omega_{jk} S_{jk}\right) + C_3 \frac{\nu_t}{e} \left(\Omega_{ik} \Omega_{jk} - \frac{1}{3} \Omega_{kl} \Omega_{kl} \delta_{ij}\right) \\
&+ C_4 \frac{\nu_t \, e}{(\epsilon^*)^2} \left(S_{kl} \Omega_{lj} + S_{kj} \Omega_{li}\right) S_{kl} \\
&+ C_5 \frac{\nu_t \, e}{(\epsilon^*)^2} \left(\Omega_{il} \Omega_{lm} S_{mj} + S_{il} \Omega_{lm} \Omega_{mj} - \frac{2}{3} S_{lm} \Omega_{mn} \Omega_{nl} \delta_{ij}\right) \\
&+ \frac{\nu_t \, e}{(\epsilon^*)^2} \left(C_6 S_{ij} S_{kl} S_{kl} + C_7 S_{ij} \Omega_{kl} \Omega_{kl}\right) \quad \text{(see next slide)}
\end{aligned}
$$

Low Re_t ASM Contd - L27($\frac{8}{19}$)

$$
\begin{aligned}
\Omega_{ij} &= (\partial u_i/\partial x_j - \partial u_j/\partial x_i) \qquad S_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i) \\
\mu_t &= f_\mu \, C_D^* \, e^2/\epsilon^* \;\rightarrow\; f_\mu = 1 - \exp\left[-(\frac{Re_t}{90})^{0.5} - (\frac{Re_t}{90})^2\right] \\
C_D^* &= 0.3 \times (1 + 0.35 \left\{\max(\overline{S}, \overline{\Omega})\right\}^{1.5})^{-1} \\
&\times \left[1 - \exp\left\{-\frac{0.36}{\exp(-0.75\max(\overline{S}, \overline{\Omega}))}\right\}\right] \\
\overline{S} &= (e/\epsilon^*)\sqrt{0.5\, S_{ij}\, S_{ij}} \qquad \overline{\Omega} = (e/\epsilon^*)\sqrt{0.5\,\Omega_{ij}\,\Omega_{ij}}
\end{aligned}
$$

Constants are: C_1 = -0.1, C_2 = 0.1, C_3 = 0.26, C_4 = - 10 $(C_D^*)^2$, C_5 = 0, C_6 = - 5 $(C_D^*)^2$ and C_7 = 5 $(C_D^*)^2$. The model is tested for very complex strain fields - swirling flows, curved channels and jet-impingement on a wall (Craft T. J., Launder B. L. and Suga K, IJHFF, 17(12), p 108, 1996)

Scalar Transport - L27($\frac{9}{19}$)

From Lecture 21,

$$\rho_m\, c_{pm}\, \left[\frac{\partial \hat{T}}{\partial t} + \frac{\partial \hat{u}_j\, \hat{T}}{\partial x_j}\right] = -\frac{\partial \hat{q}_j}{\partial x_j} + \mu\, \hat{\Phi}_v \quad \text{(Instantaneous)}$$

$$\rho_m\, c_{pm}\, \left[\frac{\partial T}{\partial t} + \frac{\partial u_j\, T}{\partial x_j}\right] = -\frac{\partial}{\partial x_j}\left(-k_m\, \frac{\partial T}{\partial x_j} + \rho_m\, c_{pm}\, \overline{u_j'\, T'}\right) + \mu_{eff}\, \Phi_v + \rho_m\, \epsilon \quad \text{(Time averaged)}$$

$\rho_m\, c_{pm}\, \overline{u_j'\, T'}$ must be obtained from

1. Eddy Diffusivity model, or
2. Transport Eqn for $\overline{u_j'\, T'}$

Eddy Diffusivity model - L27($\frac{10}{19}$)

1. Analogous to μ_t, we define Turbulent thermal conductivity k_t so that

$$-\overline{u_i'\, T'} = (\frac{k_t}{\rho\, c_p})\, \frac{\partial T}{\partial x_i} = \alpha_t\, \frac{\partial T}{\partial x_i} = \frac{\nu_t}{Pr_T}\, \frac{\partial T}{\partial x_i}$$

where Pr_T = Turbulent Prandtl number *simeq* 0.9 when Re_t is high.

2. Hence, energy Eqn will read as

$$\frac{D\, T}{D\, t} = \frac{\partial}{\partial x_k}\, \left\{ (\frac{\nu}{Pr} + \frac{\nu_t}{\sigma_t})\, \frac{\partial T}{\partial x_k} \right\} + \frac{Q_{gen}}{\rho\, c_p}$$

where $Q_{gen} = \mu_{eff}\, \Phi_v + \rho_m\, \epsilon$. Usually, $\rho_m\, \epsilon << \mu_{eff}\, \Phi_v$.

Comments on EDM - L27($\frac{11}{19}$)

1. The model is very convenient because ν_t is obtained from mixing length, or one- or two-eqn models and Pr_T is an absolute constant
2. The disadvantage is that $\alpha_t = 0$ where $\nu_t = 0$. In several flows, significant temperature gradients and hence heat transfer exist in regions where $\nu_t = 0$.
3. Like ν_t, α_t is also isotropic . But, measurement of decay of non-axi-symmetric temperature profiles in a fully developed turbulent flow in a pipe suggests that the ratio of tangential to radial diffusivities ($\alpha_{t,\theta}/\alpha_{t,r}$) $>>$ 1 near the wall.
4. Therefore, in general, $\overline{u_i' T'}$ must be obtained directly from its differential transport equation.

$\overline{u_i' T'}$ Eqn - L27($\frac{12}{19}$)

Eqn for $\overline{u_i' T'}$ is derived by multiplying Eqn for $\hat{T}$ by u_i' and Eqn for $\hat{u}_i$ by T' - addition and time-averaging gives .

$$
\begin{aligned}
\frac{\partial \overline{u_i' T'}}{\partial t} + u_k \frac{\partial \overline{u_i' T'}}{\partial x_k} \quad = \quad & - \left[\overline{u_i' u_k'} \frac{\partial T}{\partial x_k} + \overline{u_k' T'} \frac{\partial u_i}{\partial x_k} \right] \\
& \qquad\qquad \{P_T\} \\
& - \frac{\partial}{\partial x_k} \left[\overline{u_i' u_k' T'} + \frac{\overline{p' T'}}{\rho} \delta_{ik} - \alpha \frac{\partial \overline{u_i' T'}}{\partial x_k} \right] \\
& \qquad\qquad \{D_T\} \\
& + \overline{\frac{p'}{\rho} \left\{ \frac{\partial T'}{\partial x_i} \right\}} - (\nu + \alpha) \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial T'}{\partial x_k}} \\
& \qquad \{RD_T\} \qquad\qquad \{Dis_T\}
\end{aligned}
$$

Modeling $\overline{u_i' T'}$ Eqn - L27($\frac{13}{19}$)

1. Like PS_{ij}, Redistribution term RD_T is modeled as

$$
\begin{aligned}
RD_T &= -C_{T1}\,\frac{\epsilon}{e}\,\overline{u_i'\,T'} + C_{T2}\,\overline{u_k'\,T'}\,\frac{\partial u_i}{\partial x_k} \\
&= -\,0.5\,\frac{\epsilon}{e}\,\overline{u_n'\,T'}\,\frac{e^{3/2}}{\epsilon\,L_B} \quad (\text{ for Pr } > 1) \\
&= -\left\{C_{T1} + 0.5\,(\frac{Pr+1}{Pr})\right\}\,\frac{\epsilon}{e}\,\overline{u_i'\,T'} \quad (\text{ for Pr } << 1)
\end{aligned}
$$

2. At high Re_t or (Peclet), the task of Destruction is performed by RD_T . Hence, $Dis_T = 0$.
3. In the diffusion term , effect of p' is either neglected or taken to be 0.2 $\times\overline{u_i'\,u_k'\,T'}$ where

$$
-\overline{u_i'\,u_k'\,T'} = C_T\,\frac{e}{\epsilon}\,\left[\overline{u_j'\,u_k'}\,\frac{\partial\overline{u_i'\,T'}}{\partial x_j} + \overline{u_i'\,u_k'}\,\frac{\partial\overline{u_j'\,T'}}{\partial x_j}\right]
$$

Solving $\overline{u_i' T'}$ Eqn - L27($\frac{14}{19}$)

1. The model constants are: C_{T1} = 3.6, C_{T2} = 0.266 and C_T = 0.11.
2. Required correlations are taken as

$$-\overline{u_i' T'} = \frac{\nu_t}{Pr_T}\frac{\partial T}{\partial x_i} \quad \text{and} \quad -\overline{u_i' u_j'} = \nu_t S_{ij}$$

3. ν_t is determined from e and ϵ Eqns
4. For complete range of Prandtl numbers, Pr_T is modeled as

$$Pr_T = 0.85 + 0.0309 \left\{\frac{Pr + 1}{Pr}\right\}$$

Algebraic Flux Model - L27($\frac{15}{19}$)

1. Eqn for scalar fluctuations is derived as

$$\frac{D\,\overline{T'^2}/2}{Dt} = -\frac{\partial}{\partial x_i}\left[\frac{\overline{u_i'\,T'^2}}{2} - \alpha\frac{\partial}{\partial x_i}\left\{\frac{\overline{T'^2}}{2}\right\}\right] - \overline{u_i'\,T'}\,\frac{\partial T}{\partial x_i} - \alpha\,\overline{\left(\frac{\partial T'}{\partial x_i}\right)^2}$$

where $\alpha\,\overline{(\frac{\partial T'}{\partial x_i})^2} = \epsilon_T \propto \frac{e}{\epsilon}\,\overline{T'^2}$

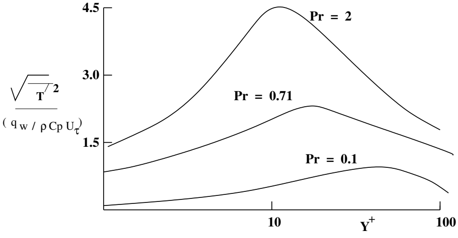
2. The AFM is derived from

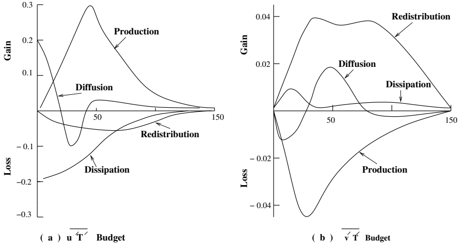
$$\frac{D\,\overline{u_i'\,T'}}{D\,t} - \mathrm{Diff}\,(\overline{u_i'\,T'}) = \left[\frac{(P-\epsilon)_e + (P-\epsilon)_{\overline{T'^2}}}{2}\right]\frac{\overline{u_i'\,T'}}{e\,\sqrt{\overline{T'^2}}}$$

$$\overline{T'^2} = C_T'\,\frac{e}{\epsilon}\,\overline{u_i'\,T'}\,\frac{\partial T}{\partial x_k} \quad \text{prod = diss assumed}$$

where $C_T' \simeq 1.6$ for $Pr \geq 1$.

Evidence from DNS - Pipe flow - L27($\frac{16}{19}$)





1. Mean T profiles for pipe flow agreed with DNS
2. Location of peak $\overline{T'^2}$ shows that production shifts towards larger y^+ as Pr decreases.

1. $\overline{u'T'}$ budget is similar to e-budget
2. $\overline{v'T'}$ budget resembles $\overline{u'v'}$ budget justifying Eddy Diff model for this case.

Combustion and Turbulence - 1 - L27($\frac{17}{19}$)

1. In Combustion it is necessary to solve differential eqns for all participating species k.

$$\frac{\partial(\rho_m\,\omega_k)}{\partial t} + \frac{\partial(\rho_m\,u_j\,\omega_k)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\rho_m\,D_{eff}\,\frac{\partial \omega_k}{\partial x_j}\right] + R_k$$

   where R_k = rate of species generation/consumption.
   $D_{eff} = \nu/Sc + \nu_t/SC_t$ and $SC_t \simeq 0.9$.
2. The simplest postulate is called the *Simple Chemical Reaction* (SCR) that is written as
   1 kg of Fuel $+ R_{st}$ kg of Oxidant $= (1 + R_{st})$ kg of Product
   There are only three species Fuel, Oxidant air and Products and R_{st} = $(A/F)_{stoich}$
3. $R_{ox} = R_{st} \times R_{fu}$ and $R_{pr} = -(1 + R_{st}) \times R_{fu}$. In laminar flow

$$R_{fu} = -\,A\,exp\,(\frac{-\,E}{R_u\,T})\,\omega_{fu}^{m}\,\omega_{ox}^{n} \quad \text{(A and E are fuel-specific)}$$

Combustion and Turbulence - 2 - L27($\frac{18}{19}$)

1. In turbulent combustion, however, it is observed that outer edges of flames are very intermittent and jagged.
2. Experimentally it is observed that even if time-averaged $\overline{\omega}_{fu}$ and $\overline{\omega}_{ox}$ are high, R_{fu} rates are not as high as would be expected from the Arrhenius formula
3. This is because, the fuel and oxidant at a given point are *present at different times*. Clearly, therefore, *time scales* of chemical reaction and turbulence are important. These are characterised by S_L / u'_{rms} where S_L is the laminar flame speed of the fuel.
4. These ideas are captured[3] in

$$R_{fu} = -\, C_{ebu}\, \rho_m \sqrt{\overline{(\omega'_{fu})^2}}\, \frac{\epsilon}{e} \simeq -\, C_{ebu}\, \rho_m\, \overline{\omega_{fu}}\, \frac{\epsilon}{e}$$

[3] Spalding D. B. *Development of Eddy-Breakup Model of Turbulent Combustion*, 16th Symposium on Combustion, p 1657, 1976

Combustion and Turbulence - 3 - L27($\frac{19}{19}$)

1. In practical computing, the applicability of the EBU has been enhanced by the following variant

$$R_{fu} = -\rho_m \frac{\epsilon}{e} \, \text{Min} \left\{ A \, \overline{\omega}_{fu} , \frac{A}{R_{st}} \, \overline{\omega}_{ox} , \frac{A^{'}}{1 + R_{st}} \, \overline{\omega}_{prod} \right\}$$

   where A = 4 and $A^{'} \simeq 2$.

2. In the next lecture, we shall discuss tow important aspects of turbulent flows: (a) Laminar-to-Turbulent Transition and (b) Effect of Wall Roughness