NPTEL Video Course
Advanced Complex Analysis – Part 2: Singularity at Infinity, Infinity as a Value, Compact Spaces of Meromorphic Functions for the Spherical Metric and Spherical Derivative, Local Analysis of Normality, Theorems of Marty-Zalcman-Montel-Picard-Royden-Schottky

http://nptel.ac.in/syllabus/111106094/
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Mid-Course Exam (Syllabus: Units 1 to 8) Time: Two Hours Maximum Marks: 40

1. State the generalised version of Liouville’s theorem. 2 marks

2. Consider the function
\[ f(z) = \frac{z^2 - 2z + 3}{z^3 + 1}. \]
   a) What kind of a singular point is \( \infty \) for \( f \)? Why?
   b) Write out the singular (principal) and analytic parts of \( f \) at \( \infty \).
   c) Verify the Residue Theorem for the extended complex plane for \( f \). 7 marks

3. Show that \( f_n(z) = z^{-n} \) converges normally to \( \infty \) in the unit disc \( |z| < 1 \). Is the convergence uniform? Justify your answer. 5 marks

4. Can a sequence of holomorphic (analytic) functions converge normally in the spherical metric to a strictly meromorphic function? Why? 2 marks

5. What kind of singularity does \( f(z) = e^z \) have at \( \infty \)? Why? 3 marks

6. A function \( f(z) \) has an isolated singularity at \( z_0 \). Given that \( f \) is a one-to-one mapping in a neighborhood of \( z_0 \), what kind of singularity can \( z_0 \) be? Why? 3 marks

7. State the Casorati-Weierstrass Theorem. Show that the only one-to-one entire functions onto the complex plane are of the form \( f(z) = az + b, a \neq 0, b \in \mathbb{C} \). 6 marks

8. Let \( f(z) = (z^2 + 1)^{-1} \).
   a) Find the spherical derivatives \( f^\#(0) \) and \( f^\#(i) \).
   b) Identify the extended complex plane with the Riemann sphere under the stereographic projection. Find the arc length of \( f(\{z : |z| = 1\}) \). 7 marks

9. Let \( f(z) \) have a pole at \( z_0 \). Prove that \( f^\#(z_0) = (1/f)^\#(z_0) \). 5 marks