

NPTEL Video Course  
Advanced Complex Analysis – Part 2: Singularity at Infinity, Infinity as a  
Value, Compact Spaces of Meromorphic Functions for the Spherical Metric  
and Spherical Derivative, Local Analysis of Normality, Theorems of  
Marty-Zalcman-Montel-Picard-Royden-Schottky

<http://nptel.ac.in/syllabus/111106094/>

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End-Course Exam (Syllabus: All Units)      Time: Three Hours      Maximum Marks: 60

1. If  $f(z)$  is defined for  $|z| > R > 0$ , define when  $\lim_{z \rightarrow \infty} f(z) = \infty$ . **2 marks**
2. How will you treat a meromorphic function on a domain as a continuous function on that domain? Why? **2 marks**
3. What is the fundamental point of difference, in studying spaces of functions with respect to convergence, between continuous functions and analytic functions? **2 marks**
4. Show that the spherical metric on the extended complex plane is invariant under the inversion  $z \mapsto z^{-1}$ . **2 marks**
5. State and prove the Casorati-Weierstrass Theorem. **5 marks**
6. State Marty's Theorem. Explain why it is stronger than its holomorphic avatar viz., Montel's theorem. **6 marks**
7. Find  $f(0 < |z| < 10^{-2014})$  if  $f(z) = e^{1/z} + e^{-1/z}$ . **3 marks**

8. Consider the family

$$\mathcal{F} := \{f_\epsilon(z) = \frac{z}{z + \epsilon}; 0 < \epsilon \leq 1\}.$$

- a) Compute the spherical derivatives of the functions of  $\mathcal{F}$ .
- b) Check  $\mathcal{F}$  for normality at infinity.
- c) Does  $\mathcal{F}$  have a non-normal point? Justify your answer.

**10 marks**

9. State and prove Zalcman's Lemma. **10 marks**
10. State the Fundamental Normality Criteria (Fundamental Normality Tests) of Montel for meromorphic and for analytic functions on a domain. **2 marks**
11. Show that the family of univalent (one-to-one) analytic functions on the open unit disc that never vanish is a normal family. **6 marks**

12. Show that a family of analytic functions  $f$  on a domain satisfying

$$|f'| \leq |f|^3$$

is normal.

**3 marks**

13. State and prove Schottky's Theorem. Explain where each of the hypotheses of the theorem have been used in the proof.

**7 marks**