Lecture 7: Moebius Transformations Make Up Fundamental Groups of Riemann Surfaces

The study of general Riemann surfaces is facilitated by the study of covering spaces.

Basic Assumption: All spaces $X$ in the forthcoming lectures are second countable and Hausdorff, unless stated otherwise.

Definition 1 A map $p : \tilde{X} \rightarrow X$, (where $\tilde{X}$, $X$ are arcwise connected and locally arcwise connected) is called a covering map if:

- $p$ is continuous and surjective;
- Every point $x \in X$ is contained in an admissible open neighbourhood $U$, i.e., $p^{-1}(U) = \bigcup_{\alpha \in I} V_{\alpha}$, $V_{\alpha} \subset \tilde{X}$ open, such that $p|_{V_{\alpha}} : V_{\alpha} \rightarrow U$ is a homeomorphism.

$\tilde{X}$ is called the covering space and $p$ is called the covering map.

7.1 Recall

$C_w = \text{the cylinder } \simeq \mathbb{C}^* \simeq \Delta^* \simeq \Delta_r$, i.e. all these four sets are homeomorphic as topological spaces. We have the quotient map $\pi_w : \mathbb{C} \rightarrow C_w = \mathbb{C}/\mathbb{Z}.T_w$, which is holomorphic and the different (non-zero) $w$-s provide the same Riemann surface structure up to isomorphism. We saw in the previous lecture that $\pi_w$ has all the properties of a covering map. Also we have $C_w$ as a quotient of $\mathbb{C}$ by a certain subgroup of automorphisms of $\mathbb{C}$ ($\mathbb{Z}.T_w$ are M"obius transformations). This is true in the case of a general universal covering space as well!

Examples of covering spaces:

1. Cylinders: $\pi_w : \mathbb{C} \rightarrow C_w = \mathbb{C}/\mathbb{Z}.T_w$. Here the Riemann surface structure is isomorphic to that on $\mathbb{C}^*$.

2. Tori: $\pi_{w_1,w_2} : \mathbb{C} \rightarrow T_{w_1,w_2} = \mathbb{C}/(\mathbb{Z}.T_{w_1} \times \mathbb{Z}.T_{w_2})$. The set of isomorphism classes of such structures is bijective to $\mathbb{C}$, and this set naturally acquires a Riemann surface structure which is just (isomorphic to) the complex plane.
Covering space theory helps us distinguish (or classify) Riemann surface structures.

Suppose $X$ is a Riemann surface. Then the underlying topological space is connected, arcwise connected, locally arcwise connected and locally simply connected. For such a space it can be shown that we can get a covering $p : \tilde{X} \to X$, with $\tilde{X}$ simply connected. Such a covering space is called a **Universal covering space** for $X$ and is uniquely determined up to isomorphism.

Any covering space (not necessarily universal) of a Riemann surface also inherits a Riemann surface structure, defined uniquely up to an isomorphism, such that the covering map becomes holomorphic. For, if $X$ is any Riemann surface, $\tilde{X}$ is any covering space and $p : \tilde{X} \to X$ is the covering map, then $p$ being a local homeomorphism, given $x \in \tilde{X}$, we have a neighbourhood $V$ around it such that $p(V) = U$ an open subset of $X$, and $p|_V : V \to U$ is a homeomorphism (we get this from the definition of covering space). Since $X$ is a Riemann surface, $X$ has charts locally, and since $\tilde{X}$ is locally homeomorphic to $X$, we can transport these charts to $\tilde{X}$. In this way, we can make $\tilde{X}$ into a Riemann surface. So, if we have a covering space of a Riemann surface, we can make the covering space into a Riemann surface as well, and this Riemann surface structure is in fact determined uniquely up to an isomorphism.

What happened in the case of the cylinder $\pi_w : \mathbb{C} \to C_w$ or the torus $\pi_{w_1, w_2} : \mathbb{C} \to \mathbb{T}_{w_1, w_2}$? Both are covering maps, but to begin with $C_w$ and $\mathbb{T}_{w_1, w_2}$ were not Riemann surfaces. We wanted to give the cylinder and the torus the structure of a Riemann surface, and we got the Riemann surface structure on the cylinder and the torus because of the covering space which is a Riemann surface; i.e. because the covering maps $\pi_w$ and $\pi_{w_1, w_2}$ are local homeomorphisms, we were able to get charts and define a Riemann surface structure on the target space for which the covering maps became holomorphic. So what really happens in these two cases is that we have a covering space situation as follows: on top is a Riemann surface ($\mathbb{C}$), and what we get below (as the Riemann surface we wanted) is a quotient of the space on top ($\mathbb{C}$) by a subgroup of automorphisms of the space on top (in this case, they are Möbius transformations, which are automorphisms of $\mathbb{C}$). That is, in the case of the cylinder or the torus, we are obtaining a Riemann surface structure on the space below.

We have a topological space which is the base space of a covering, and since the space on top is already a Riemann surface, the space below also becomes a Riemann surface. What we saw earlier was the converse, i.e., if we start with a Riemann surface and we take any covering space of that Riemann surface, then the covering space becomes a Riemann surface so that the covering map becomes holomorphic.
Essentially what this means is that given a covering space situation, if we have a Riemann surface structure on the top, we can push it to the bottom (in nice cases); further the converse always holds.

Now in particular, if we take the universal covering space of the Riemann surface, then the universal covering space also becomes a Riemann surface. But the universal covering space is simply connected. Thus by the Uniformization Theorem, it has to be the complex plane $\mathbb{C}$ or the unit disc $\Delta$ (equivalently, the upper half plane $\mathbb{U}$) or the Riemann sphere $\mathbb{P}_1^{\mathbb{C}}$. This means that every Riemann surface is obtained from these three known Riemann surfaces by going modulo a certain group of automorphisms (Möbius transformations in this case).

We have been talking about the covering space. The next obvious question we ask is: where does the fundamental group come into the picture?

- The fundamental group of $C_w$ is $\mathbb{Z}$. The fibre over any point in $C_w$ is bijective to $\mathbb{Z}$. $\mathbb{Z}$ is also isomorphic to the group of translations going modulo which we get the Riemann surface below. $\mathbb{C}/\Pi_1(C_w) = \mathbb{C}/\mathbb{Z} \cong \mathbb{C}/\mathbb{Z}.T_w$.

- The fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$, and we have a similar situation as above.

The situation above is one that holds in general: in the case of a universal covering $p : \tilde{X} \rightarrow X$, the fibres $p^{-1}(x)$ are bijective to the (first) fundamental group $\Pi_1(X)$, which can also be identified with a subgroup of automorphisms of the covering space; moreover $X$ is precisely the quotient of $\tilde{X}$ by this subgroup. Therefore, $\tilde{X}/\Pi_1(X) \cong X$ as Riemann surfaces and the covering map $p : \tilde{X} \rightarrow X$ is just the quotient map $\tilde{X} \rightarrow \tilde{X}/\Pi_1(X,x)$. Hence, any Riemann surface structure is a quotient of $\mathbb{C}$ or $\Delta$ or $\mathbb{P}_1^{\mathbb{C}}$ by a subgroup of automorphisms (Moebius transformations) isomorphic to the fundamental group of the Riemann surface.

In conclusion, to study any Riemann surface, we need to study subgroups of automorphisms of $\mathbb{C}$ or $\Delta$ or $\mathbb{P}_1^{\mathbb{C}}$, i.e., subgroups of Möbius transformations. We recall that:

- $Aut_{Hol}(\mathbb{P}_1^{\mathbb{C}}) = PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\mathbb{Z}_2$,
- $Aut_{Hol}(\mathbb{U}) = PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\mathbb{Z}_2$,
- $Aut_{Hol}(\mathbb{C}) = P\Delta(2,\mathbb{C})$, i.e. the upper triangular elements of $PSL(2,\mathbb{C})$. 