Lecture 6: Riemann Surface Structures on Cylinders and Tori via Covering Spaces

Consider the following four sets:

- The cylinder \( \mathbb{C}/\mathbb{Z} \) where \( \mathbb{Z} \) is thought of as the subgroup of translations by integer multiples of a fixed non-zero complex number;
- The punctured plane \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \);
- The punctured unit disc \( \Delta^* = \Delta \setminus \{0\} \), where, \( \Delta = \{z : |z| < 1\} \) and
- The annulus \( \Delta_r = \{z \in \mathbb{C} | r < |z| < 1\} \).

Each of the above topological spaces is homeomorphic to the others. It is easy to see that \( \mathbb{C}^* \), \( \Delta^* \) and \( \Delta_r \) are homeomorphic to one another. One way to see that \( \mathbb{C}^* \) is homeomorphic to the cylinder \( \mathbb{C}/\mathbb{Z} \) is the following. Consider the sphere \( S^2 \). We can see from the above image that there is a homeomorphism from the cylinder \( \mathbb{C}/\mathbb{Z} \) to \( S^2 \setminus \{N, S\} \) where \( N, S \) denote respectively the “north-” and “south poles” of the sphere. This homeomorphism is obtained by sending each point \( P \) on the cylinder to the point \( Q \) on \( S^2 \), where the line joining \( P \) with the centre of \( S^2 \) pierces \( S^2 \) at \( Q \). Thereby \( S^2 \setminus \{N, S\} \) becomes homeomorphic to \( \mathbb{C}/\mathbb{Z} \). We know that \( S^2 \setminus \{N\} \cong \mathbb{C} \) by the stereographic projection from \( N \). So \( S^2 \setminus \{N, S\} \cong \mathbb{C} \setminus \{0\} \) since under this stereographic projection the south pole \( S \) corresponds to the origin. Hence we now have \( \mathbb{C}/\mathbb{Z} \cong S^2 \setminus \{N, S\} \cong \mathbb{C}^* \). Recall the following theorem that we stated in the previous lecture.
**Theorem 1** For $z_0 \neq 0$ in $\mathbb{C}$ consider the Riemann surface structure $C_{z_0}$ on the cylinder $C = \mathbb{C}/\mathbb{Z}.T_{z_0}$ and the quotient map $\pi_{z_0}$ which is holomorphic:

$$\pi_{z_0} : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}.T_{z_0} : z \mapsto \text{equivalence class}(z) = \{z + nz_0 : n \in \mathbb{Z}\}.$$  

The set $\{C_w : w \in \mathbb{C}\setminus\{0\}\} \mod$ isomorphism of Riemann surfaces is a singleton and is represented by the Riemann surface structure on $\mathbb{C}^\ast$.

We now note that it is impossible to find a non-constant (bi)holomorphic map from $\mathbb{C}^\ast$ to $\Delta^\ast$ or to $\Delta_r$. For if we do have such a holomorphic map $f$ from $\mathbb{C}^\ast$ to say $\Delta^\ast$, then the map in a deleted neighbourhood of the origin would be bounded as the target set is bounded. By Riemann’s theorem on removable singularities $f$ would extend to a holomorphic map from all of $\mathbb{C}$. Recall that the Riemann theorem on removable singularities is the following. Let $D \subset \mathbb{C}$ be an open subset of the complex plane, $a \in D$ and $f$ a holomorphic function defined on $D \setminus \{a\}$. The following are then equivalent:

- $f$ is holomorphically extendable to $a$.
- $f$ is continuously extendable to $a$.
- There exists a deleted neighborhood of $a$ on which $f$ is bounded.

To sum up we have an entire function $f$ that is bounded, which by Liouville’s theorem has to be constant. All this shows we cannot have a non-constant (bi)holomorphic map from $\mathbb{C}^\ast$ to $\Delta^\ast$. A similar argument shows that there cannot be non-constant holomorphic maps from the punctured plane to $\Delta_r$. Hence the natural Riemann surface structures on $\Delta^\ast$ and $\Delta_r$ induced from the complex plane are certainly going to be Riemann surface structures on the cylinder different from that induced on $\mathbb{C}^\ast$. In fact, for different values of $r$, the corresponding Riemann surfaces $\Delta_r$ are not biholomorphic to one another and moreover no $\Delta_r$ can be biholomorphic to $\Delta^\ast$.

**Theorem 2** The set of isomorphism classes of all Riemann surface structures on the cylinder is given as the disjoint union of three sets, two of these being singletons and the third a one-real-parameter family as follows:

$$\{[\mathbb{C}^\ast]\} \amalg \{[\Delta^\ast]\} \amalg \{[\Delta_r] : r \in (0,1)\}.$$  

Here $[X]$ denotes the (biholomorphic or conformal or analytic) isomorphism class of the Riemann surface $X$.

How do we distinguish between the Riemann surfaces occurring in the above three sets? This is a question about classifying Riemann surfaces up to isomorphism and Covering Space Theory is the tool that is used to answer it.
Definition 1 Let $X$ and $\tilde{X}$ be topological spaces. Assume $X$ and $\tilde{X}$ are pathwise and locally pathwise connected. A map $p : \tilde{X} \rightarrow X$ is called a covering map if:

- $p$ is continuous and surjective, and
- given $x \in X$, $\exists$ an open set $U$, $x \in U$, such that $p^{-1}(U) = \coprod_{\alpha \in I} V_{\alpha}$, where $V_{\alpha} \subset \tilde{X}$ is open and $p|_{V_{\alpha}} : V_{\alpha} \rightarrow U$ is a homeomorphism.

Observation:
We do not have non-constant holomorphic maps $\mathbb{C}^* \rightarrow \Delta^*$ or $\Delta^* \rightarrow \Delta_r$. However, we do have holomorphic quotient maps $\mathbb{C} \rightarrow \mathbb{C}^*$ or $\mathbb{C} \rightarrow C_w$ which are covering maps.

\[
\begin{array}{c}
\mathbb{C} \\
\downarrow \pi_w \\
\mathbb{C}/\mathbb{Z}.T_w
\end{array}
\]

$\pi_w$ is surjective and continuous. $\pi_w|_{D} : D \rightarrow \pi_w(D)$ is a homeomorphism (refer to the following image). $\pi_w^{-1}(\pi_w(D)) = \bigcup_{n \in \mathbb{Z}} (D + nw)$ which is a disjoint union of open sets and $\pi_w|_{D+nw} : D + nw \rightarrow \pi_w(D + nw) = \pi_w(D)$ is a homeomorphism. Hence, $\pi_w$ is a covering map. Consider next the case of the Riemann surface on a torus we obtained by fixing $w_1$, $w_2 \in \mathbb{C} \setminus \{0\}$ with $w_1/w_2 \notin \mathbb{R}$. The quotient map is

\[
\pi_{w_1,w_2} : \mathbb{C} \rightarrow \mathbb{C}/(\mathbb{Z}.T_{w_1} + \mathbb{Z}.T_{w_2}) = T_{w_1,w_2}.
\]

It is easy to check that $\pi_{w_1,w_2}$ is a covering map as well.