Statistical Inference
Test Set 4

1. Let $X_1, X_2, ..., X_n$ be a random sample from a $N(\mu, \sigma^2)$ population. Find UMVUEs of the signal to noise ration $\frac{\mu}{\sigma}$ and quantile $\mu + b\sigma$, where b is any given real.
2. Let $X_1, X_2, ..., X_n$ be a random sample from a $N(\mu, 1)$ population. Find a UMVUE of cdf $\Phi(x-\theta)$, where Φ denotes the cdf of a standard normal variable.
3. Let $X \sim Bin\,(n, p)$, where p is known. Find UMVUEs of p^2 and $Var(X)$.
4. Let $X_1, X_2, ..., X_n$ be a random sample from a $\mathrm{P}(\lambda)$ population. Find a UMVUE of $g(\lambda) = P(X_1 \le 1) = (1+\lambda)\, e^{-\lambda}$.
5. Let $X_1, X_2, ..., X_n$ be a random sample from a $Geo\,(p)$ distribution. Find a UMVUE of $P(X_1 = 1) = p$.
6. Let $X_1, X_2, ..., X_n$ be a random sample from a Gamma (p, λ) population, where p is known. Find a UMVUE of λ^m and λ^{-r}, where m and r are positive integers.
7. Let $X_1, X_2, ..., X_n$ be a random sample from an exponential population with the density $f(x) = e^{\mu - x},\ x > \mu,\ \mu \in \mathbb{R}.$ Find UMVUEs of μ and μ^2.
8. Let $X_1, X_2, ..., X_n$ be a random sample from $Exp\,(\mu, \sigma)$ population. Find UMVUEs of a quantile $\mu + b\sigma$ and reliability function $R(t) = P(X_1 > t)$.
9. Let $X_1, X_2, ..., X_n$ be a random sample from a $N(0, \sigma^2)$ population. Find the best scale equivariant estimators of σ^2 and σ with respect to scale invariant loss functions.
10. Let $X_1, X_2, ..., X_n$ be a random sample from $Exp\,(\mu, \sigma)$ population. Find best affine equivariant estimator of $\theta = \mu + \eta\sigma$ with respect to an affine invariant loss function.
11. Let $X_1, X_2, ..., X_n$ be a random sample from an exponential population with density $f(x \mid \theta) = \theta e^{-\theta x},\ x > 0, \theta > 0.$ Find Bayes estimator of θ with respect to the prior $g(\theta) = e^{-\theta}, \theta > 0.$ The loss functions are $L_1(\theta, a) = (\theta - a)^2$, $L_2(\theta, a) = (\theta - a)^2 / \theta^2$ and $L_3(\theta, a) = (\theta - a)^2 / a$.
12. Let $X_1, X_2, ..., X_n$ be a random sample from a $U(0, \theta)$ population. Find Bayes estimator of θ with respect to the prior $g(\theta) = \frac{\alpha\beta^\alpha}{\theta^{\alpha+1}}, \theta > \beta.$ The loss function is $L(\theta, a) = (\theta - a)^2$.

Hints and Solutions

1. As $(\overline{X}, S^2)$ is complete and sufficient, one can use Rao-Blackwell-Lehmann-Scheffe Theorem to show that estimators U_1 and U_2 as defined in Hints and Solutions in Test Set 1 are UMVUEs of $\frac{\mu}{\sigma}$ and $\mu + b\sigma$ respectively.

2. Since $\overline{X}$ is complete and sufficient, using Rao-Blackwell-Lehmann-Scheffe Theorem, we conclude that $h(\overline{X})$ is a UMVUE of $\Phi(x-\theta)$, where $h(\overline{x}) = P(X_1 \le x \mid \overline{X} = \overline{x})$. The conditional distribution of $X_1 \mid \overline{X} = \overline{x}$ is $N\left(\overline{x}, \frac{n-1}{n}\right)$. It can be then shown that $h(\overline{x}) = \Phi\left(\sqrt{\frac{n}{n-1}}(x-\overline{x})\right)$.

3. UMVUEs of p^2 and $Var(X)$ are respectively given by $T_1 = \frac{X(X-1)}{n(n-1)}$ and $T_2 = \frac{X(n-x)}{n-1}$.

4. Let $T(X_1) = 1,\quad$ if $X_1 = 0$ or 1.

   $\qquad\qquad = 0,\quad$ otherwise.

   Then $T(X_1)$ is unbiased for $g(\lambda)$. Note that $S = \sum_{i=1}^{n} X_i$ is complete and sufficient statistic. So using Rao-Blackwell-Lehmann-Scheffe Theorem, $h(S) = E(T(X_1) \mid S)$ is a UMVUE of $g(\lambda)$. Now $h(s) = P(X_1 = 0 \mid S = s) + P(X_1 = 1 \mid S = s)$.

   $$P(X_1 = 0 \mid S = s) = \frac{P(X_1 = 0, S = s)}{P(S = s)} = \frac{P(X_1 = 0, \sum_{i=2}^{n} X_i = s)}{P(S = s)}$$

   Using independence of X_1 and $\sum_{i=2}^{n} X_i$ and the additive property of Poisson distribution, we get the above expression as $\left(\frac{n-1}{n}\right)^s$. In a similar way, we get

   $P(X_1 = 1 \mid S = s) = \frac{s(n-1)^{s-1}}{n^s}$. Thus $h(S) = \frac{(n-1)^S (S+n-1)}{n^S}$.

5. Let $T(X_1) = 1,\quad$ if $X_1 = 0$

   $\qquad\qquad = 0,\quad$ otherwise.

   As in Qn. 5, $h(S)$ is a UMVUE of p, where $h(s) = P(X_1 = 0 \mid S = s)$ and $S = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic. The distribution of S is negative binomial (n, p). Proceeding as in Qn. 5, we get $h(S) = \frac{n-1}{S-1}$.

6. A complete and sufficient statistic is $T = \sum_{i=1}^{n} X_i$. Also $T \sim Gamma(np, \lambda)$. We have

$$E\left(\frac{\lceil np-m}{\lceil np} T^{-m}\right) = \lambda^m \text{ , } np > m \text{ and } E\left(\frac{\lceil np}{\lceil np+r} T^{r}\right) = \lambda^{-r}$$

7. A complete and sufficient statistic is $Y = X_{(1)}$. The density of Y is $f(y) = ne^{n(\mu-y)}, y > \mu.$ We have $E(Y) = \mu + \frac{1}{n}$ and $E(Y^2) = \mu^2 + \frac{2\mu}{n} + \frac{2}{n^2}$. Using these UMVUEs of μ and μ^2 are $Y - \frac{1}{n}$ and $Y^2 - \frac{2Y}{n}$.

8. Let $Y = X_{(1)}$ and $Z = \sum_{i=1}^{n}(X_i - Y)$. Then (Y, Z) is complete and sufficient. Also, Y and Z are independently distributed with $Y \sim Exp\left(\mu, \frac{\sigma}{n}\right)$ and $\frac{2Z}{\sigma} \sim \chi^2_{2n-2}$. Using these, UMVUEs of μ and σ are given by $d_1 = Y - \frac{Z}{n(n-1)}$ and $d_2 = \frac{Z}{n-1}$ respectively. So a UMVUE for quantile is $d_1 + bd_2$.

A UMVUE for $R(t)$ is $h(Y,Z) = P(X_1 > t \mid (Y,Z))$.

It can be seen that $h(Y,Z) = \frac{n-1}{n}\left[\max\left\{1 - \frac{t-Y}{Z}, 0\right\}\right]^{n-2}$.

9. Note that $T = \sum X_i^2$ is a complete and sufficient statistic. Also $W = \frac{T}{\sigma^2} \sim \chi^2_n$. Let the loss functions for estimating σ^2 and σ be $L_1(\sigma^2, a) = \left(\frac{\sigma^2 - a}{\sigma^2}\right)^2$ and $L_2(\sigma, b) = \left(\frac{\sigma - b}{\sigma}\right)^2$ respectively. Clearly the two estimation problems are invariant under the scale group of transformations, $G_S = \{g_c : g_c(x) = cx, c > 0\}$ on the space of $X_i s$. Under the transformation g_c, note that $\sigma^2 \to c^2\sigma^2$, $a \to c^2 a, \sigma \to c\sigma, b \to cb.$ The form of a scale equivariant estimator of σ^2 is $d_k(T) = kT$, where k is a positive constant. Minimizing the risk function of d_k with respect to k, we get $k = \frac{\sigma^2 E(T)}{E(T^2)} = \frac{n\sigma^4}{n(n+2)\sigma^4} = \frac{1}{n+2}$. Hence $\frac{T}{n+2}$ the best scale equivariant estimator of σ^2. Similarly, the form of a scale equivariant estimator of σ is $U_p(T) = pT^{1/2}$, where p is a positive constant. Minimizing the risk function of U_p with

respect to p, we get $p = \frac{\sigma E(T^{1/2})}{E(T)} = \frac{\sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right)\sigma^2}{\Gamma\left(\frac{n}{2}\right)\, n\,\sigma^2} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n+2}{2}\right)}$. So $\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n+2}{2}\right)} T^{1/2}$ is the best scale equivariant estimator of σ.

10. We follow the notation of Qn 8. Let the loss function be $L(\mu,\sigma,a) = \left(\frac{\theta - a}{\sigma}\right)^2$. The estimation problem is invariant under the affine group of transformations, $G_A = \{g_{b,c} : g_{b,c}(x) = bx + c, b > 0, c \in \mathbb{R}\}$ on the space of $X_i s$. Under the transformation $g_{b,c}$, note that $\mu \to b\mu + c$, $\sigma \to b\,\sigma, \theta \to b\theta + c, a \to ba + c, Y \to bY + c, Z \to bZ$. The form of an affine equivariant estimator of θ is $d_k(Y,Z) = Y + kZ$, where k is a constant. Minimizing the risk function of d_k with respect to k, we get $\hat{k} = \frac{E(\theta - Y)Z}{E(Z^2)} = \frac{E(\theta - Y)EZ}{E(Z^2)}$ $= \frac{\left(\mu + \eta\sigma - \mu - \frac{\sigma}{n}\right)(n-1)\sigma}{n(n-2)\sigma^2} = \frac{(n-1)\left(\eta - \frac{1}{n}\right)}{n(n-2)}$. So the best affine equivariant estimator of θ is $d_{\hat{k}}$.

11. The joint pdf of $\underline{X} = (X_1, X_2, ..., X_n)$ is $f(\underline{x} \mid \theta) = \theta^n \exp\left\{-\theta \sum_{i=1}^{n} x_i\right\}, x_i > 0, \theta > 0.$

The joint pdf of $\underline{X}$ and θ is $f^*(\underline{x},\theta) = \theta^n \exp\left\{-\theta\left(\sum_{i=1}^{n} x_i + 1\right)\right\}, x_i > 0, \theta > 0.$

The marginal density of $\underline{X}$ is then $h(\underline{x}) = \frac{\Gamma(n+1)}{(n\bar{x}+1)^{n+1}}, x_i > 0.$

Hence the posterior density of θ given $\underline{X} = \underline{x}$ is $Gamma(n+1, n\bar{x}+1)$.

Note that

$$E(\theta \mid \underline{x}) = \frac{n+1}{n\bar{x}+1}, E(\theta^2 \mid \underline{x}) = \frac{(n+1)(n+2)}{(n\bar{x}+1)^2}, \quad E\left(\frac{1}{\theta} \middle| \underline{x}\right) = \frac{n\bar{x}+1}{n}, E\left(\frac{1}{\theta^2} \middle| \underline{x}\right) = \frac{(n\bar{x}+1)^2}{n(n-1)}.$$

With respect to the loss function L_1, the Bayes estimator of θ is $E(\theta \mid \underline{X}) = \frac{n+1}{n\bar{X}+1}$.

With respect to the loss function L_2, the Bayes estimator of θ is $\frac{E\left(\frac{1}{\theta} \middle| \underline{X}\right)}{E\left(\frac{1}{\theta^2} \middle| \underline{X}\right)} = \frac{n-1}{n\bar{X}+1}$.

With respect to the loss function L_3, the Bayes estimator of θ is $\left\{E(\theta^2 \mid \underline{X})\right\}^{1/2} = \frac{\sqrt{(n+1)(n+2)}}{(n\bar{X}+1)}$.

12. The joint pdf of $\underline{X} = (X_1, X_2, ..., X_n)$ is $f(\underline{x} \mid \theta) = \dfrac{1}{\theta^n}, 0 < x_{(1)} < \cdots\cdots < x_{(n)} < \theta.$

The joint pdf of $\underline{X}$ and θ is $f^*(\underline{x}, \theta) = \dfrac{\alpha\beta^\alpha}{\theta^{n+\alpha+1}}, \theta > \max\{\beta, x_{(n)}\}.$

The marginal density of $\underline{X}$ is then $h(\underline{x}) = \dfrac{\alpha\beta^\alpha}{(n+\alpha)\left[\max\{\beta, x_{(n)}\}\right]^{n+\alpha}}, x_{(n)} > 0.$

Hence the posterior density of θ given $\underline{X} = \underline{x}$ is

$$g^*(\theta \mid \underline{x}) = \frac{(n+\alpha)\left[\max\{\beta, x_{(n)}\}\right]^{n+\alpha}}{\theta^{n+\alpha+1}}, \theta > \max\{\beta, x_{(n)}\}.$$

With respect to the loss function L, the Bayes estimator of θ is $E(\theta \mid \underline{X}) = \dfrac{n+\alpha}{n+\alpha-1}\max\{\beta, X_{(n)}\}$.