1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ population. Find UMVUEs of the signal to noise ratio $\frac{\mu}{\sigma}$ and quantile $\mu + b\sigma$, where $b$ is any given real.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, 1)$ population. Find a UMVUE of $\Phi(x - \theta)$, where $\Phi$ denotes the cdf of a standard normal variable.

3. Let $X \sim \text{Bin}(n, p)$, where $p$ is known. Find UMVUEs of $p^2$ and $\text{Var}(X)$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $P(\lambda)$ population. Find a UMVUE of $g(\lambda) = P(X_1 \leq 1) = (1 + \lambda)e^{-\lambda}$.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $\text{Geo}(p)$ distribution. Find a UMVUE of $P(X_1 = 1) = p$.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Gamma $(p, \lambda)$ population, where $p$ is known. Find a UMVUE of $\lambda^m$ and $\lambda^{-r}$, where $m$ and $r$ are positive integers.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential population with the density $f(x) = e^{\mu-x}, \ x > \mu, \mu \in \mathbb{R}$. Find UMVUEs of $\mu$ and $\mu^2$.

8. Let $X_1, X_2, \ldots, X_n$ be a random sample from $\text{Exp}(\mu, \sigma)$ population. Find UMVUEs of a quantile $\mu + b\sigma$ and reliability function $R(t) = P(X_1 > t)$.

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(0, \sigma^2)$ population. Find the best scale equivariant estimators of $\sigma^2$ and $\sigma$ with respect to scale invariant loss functions.

10. Let $X_1, X_2, \ldots, X_n$ be a random sample from $\text{Exp}(\mu, \sigma)$ population. Find best affine equivariant estimator of $\theta = \mu + \eta\sigma$ with respect to an affine invariant loss function.

11. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential population with density $f(x | \theta) = \theta e^{-\theta x}, \ x > 0, \theta > 0$. Find Bayes estimator of $\theta$ with respect to the prior $g(\theta) = e^{-\theta}, \theta > 0$. The loss functions are $L_1(\theta, a) = (\theta - a)^2$, $L_2(\theta, a) = (\theta - a)^2 / \theta^2$ and $L_3(\theta, a) = (\theta - a)^2 / a$.

12. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $U(0, \theta)$ population. Find Bayes estimator of $\theta$ with respect to the prior $g(\theta) = \frac{\alpha \beta^\alpha}{\theta^{\alpha+1}}, \theta > \beta$. The loss function is $L(\theta, a) = (\theta - a)^2$. 

Hints and Solutions

1. As \((\bar{X}, S^2)\) is complete and sufficient, one can use Rao-Blackwell-Lehmann-Scheffe Theorem to show that estimators \(U_1\) and \(U_2\) as defined in Hints and Solutions in Test Set 1 are UMVUEs of \(\frac{\mu}{\sigma}\) and \(\mu + b\sigma\) respectively.

2. Since \(\bar{X}\) is complete and sufficient, using Rao-Blackwell-Lehmann-Scheffe Theorem, we conclude that \(h(\bar{X})\) is a UMVUE of \(\Phi(x - \theta)\), where \(h(\bar{x}) = P(X_i \leq x \mid \bar{X} = \bar{x})\). The conditional distribution of \(X_i \mid \bar{X} = \bar{x}\) is
\[N\left(\bar{x}, \frac{n-1}{n}\right)\]. It can be then shown that
\[h(\bar{x}) = \Phi\left(\sqrt{\frac{n}{n-1}}(x - \bar{x})\right)\].

3. UMVUEs of \(p^2\) and \(\text{Var}(X)\) are respectively given by \(T_1 = \frac{X(X - 1)}{n(n-1)}\) and \(T_2 = \frac{X(n-x)}{n-1}\).

4. Let \(T(X_i) = 1\), if \(X_i = 0\) or \(1\),
\[= 0, \quad \text{otherwise}.
\]
Then \(T(X_i)\) is unbiased for \(g(\lambda)\). Note that \(S = \sum_{i=1}^{n} X_i\) is complete and sufficient statistic.

So using Rao-Blackwell-Lehmann-Scheffe Theorem, \(h(S) = E(T(X_i) \mid S)\) is a UMVUE of \(g(\lambda)\). Now \(h(s) = P(X_i = 0 \mid S = s) + P(X_i = 1 \mid S = s)\).

\[P(X_i = 0 \mid S = s) = \frac{P(X_i = 0, S = s)}{P(S = s)} = \frac{P(X_i = 0, \sum_{i=2}^{n} X_i = s)}{P(S = s)}
\]
Using independence of \(X_i\) and \(\sum_{i=2}^{n} X_i\) and the additive property of Poisson distribution, we get the above expression as \(\left(\frac{n-1}{n}\right)^s\). In a similar way, we get
\[P(X_i = 1 \mid S = s) = \frac{s(n-1)^{s-1}}{n^s}\]. Thus \(h(S) = \frac{(n-1)^s(S + n - 1)}{n^s}\).

5. Let \(T(X_i) = 1\), if \(X_i = 0\)
\[= 0, \quad \text{otherwise}.
\]
As in Qn. 5, \(h(S)\) is a UMVUE of \(p\), where \(h(s) = P(X_i = 0 \mid S = s)\) and \(S = \sum_{i=1}^{n} X_i\) is a complete and sufficient statistic. The distribution of \(S\) is negative binomial \((n, p)\).
Proceeding as in Qn. 5, we get \(h(S) = \frac{n-1}{S-1}\).
6. A complete and sufficient statistic is $T = \sum_{i=1}^{n} X_i$. Also $T \sim \text{Gamma}(np, \lambda)$. We have 
$$E\left(\frac{np - m}{np} T^{-m}\right) = \lambda^m, \; np > m \quad \text{and} \quad E\left(\frac{np}{np + r} T^r\right) = \lambda^{-r}$$

7. A complete and sufficient statistic is $Y = X_{(1)}$. The density of $Y$ is $f(y) = ne^{n(\mu - y)}, \; y > \mu$.
We have $E(Y) = \mu + \frac{1}{n}$ and $E(Y^2) = \mu^2 + \frac{2\mu}{n} + \frac{2}{n^2}$. Using these UMVUEs of $\mu$ and $\mu^2$ are $Y - \frac{1}{n}$ and $Y^2 - \frac{2Y}{n}$.

8. Let $Y = X_{(1)}$ and $Z = \sum_{i=1}^{n}(X_i - Y)$. Then $(Y, Z)$ is complete and sufficient. Also, $Y$ and $Z$ are independently distributed with $Y \sim \text{Exp}\left(\frac{\mu}{\sigma}, \frac{\sigma}{n}\right)$ and $\frac{2Z}{\sigma} \sim \chi_{2n-2}^{2}$. Using these, UMVUEs of $\mu$ and $\sigma$ are given by $d_1 = Y - \frac{Z}{n(n-1)}$ and $d_2 = \frac{Z}{n-1}$ respectively. So a UMVUE for quantile is $d_1 + bd_2$.
A UMVUE for $R(t)$ is $h(Y, Z) = P(X_i > t | (Y, Z))$.
It can be seen that $h(Y, Z) = \frac{n-1}{n} \left[ \max \left\{ 1 - \frac{t-Y}{Z}, 0 \right\} \right]^{n-2}$.

9. Note that $T = \sum X_i^2$ is a complete and sufficient statistic. Also $W = \frac{T}{\sigma^2} \sim \chi_n^2$. Let the loss functions for estimating $\sigma^2$ and $\sigma$ be $L_1(\sigma^2, a) = \left(\frac{\sigma^2 - a}{\sigma^2}\right)^2$ and $L_2(\sigma, b) = \left(\frac{\sigma - b}{\sigma}\right)^2$ respectively. Clearly the two estimation problems are invariant under the scale group of transformations, $G_s = \{ g_c : g_c(x) = cx, c > 0 \}$ on the space of $X_i$s. Under the transformation $g_c$, note that $\sigma^2 \rightarrow c^2 \sigma^2$, $a \rightarrow c^2 a$, $\sigma \rightarrow c \sigma$, $b \rightarrow cb$. The form of a scale equivariant estimator of $\sigma^2$ is $d_k(T) = kT$, where $k$ is a positive constant. Minimizing the risk function of $d_k$ with respect to $k$, we get $k = \frac{\sigma^2 E(T)}{E(T^3)} = \frac{n\sigma^4}{n(n+2)\sigma^4} = \frac{1}{n+2}$. Hence $\frac{T}{n+2}$ the best scale equivariant estimator of $\sigma^2$. Similarly, the form of a scale equivariant estimator of $\sigma$ is $U_p(T) = pT^{1/2}$, where $p$ is a positive constant. Minimizing the risk function of $U_p$ with
respect to $p$, we get $p = \frac{\sigma E(T^{1/2})}{E(T)} = \sqrt{2} \left( \frac{n+1}{2} \right) \frac{\sigma^2}{n \sigma^2} = \sqrt{2} \left( \frac{n+1}{2} \right) \frac{\sigma^2}{n \sigma^2}$. So $\sqrt{2} \left( \frac{n+1}{2} \right) T^{1/2}$ is the best scale equivariant estimator of $\sigma$.

10. We follow the notation of Qn 8. Let the loss function be $L(\mu, \sigma, a) = \left( \frac{\theta-a}{\sigma} \right)^2$. The estimation problem is invariant under the affine group of transformations, $G_a = \{g_{b,c} : g_{b,c}(x) = bx + c, b > 0, c \in \mathbb{R}\}$ on the space of $X_i$s. Under the transformation $g_{b,c}$, note that $\mu \rightarrow b\mu + c$, $\sigma \rightarrow b\sigma$, $\theta \rightarrow b\theta + c, a \rightarrow ba + c, Y \rightarrow bY + c, Z \rightarrow bZ$. The form of an affine equivariant estimator of $\theta$ is $d_k(Y, Z) = Y + kZ$, where $k$ is a constant. Minimizing the risk function of $d_k$ with respect to $k$, we get $\hat{k} = \frac{E(\theta - Y)Z}{E(Z^2)} - \frac{E(\theta - Y)EZ}{E(Z^2)}$

$$= \frac{(\mu + \eta \sigma - \mu - \frac{\sigma}{n})(n-1)\sigma}{n(n-2)\sigma^2} = \frac{(n-1)(\eta - \frac{1}{n})}{n(n-2)}.$$ So the best affine equivariant estimator of $\theta$ is $d_{\hat{k}}$.

11. The joint pdf of $X = (X_1, X_2, \ldots, X_n)$ is $f(x | \theta) = \theta^n \exp \left\{ -\theta \sum_{i=1}^n x_i \right\}, x_i > 0, \theta > 0$.

The joint pdf of $X$ and $\theta$ is $f^*(x, \theta) = \theta^n \exp \left\{ -\theta \left( \sum_{i=1}^n x_i + 1 \right) \right\}, x_i > 0, \theta > 0$.

The marginal density of $X$ is then $h(x) = \frac{n+1}{(nx + 1)^{n+1}}, x_i > 0$.

Hence the posterior density of $\theta$ given $\bar{x} = \frac{\sum x_i}{n}$ is $\text{Gamma}(n+1, n\bar{x}+1)$.

Note that $E(\theta | \bar{x}) = \frac{n+1}{n\bar{x} + 1}, E(\theta^2 | \bar{x}) = \frac{(n+1)(n+2)}{(n\bar{x}+1)^2}, E\left( \frac{1}{\theta} | \bar{x} \right) = \frac{n\bar{x}+1}{n}, E\left( \frac{1}{\theta^2} | \bar{x} \right) = \frac{(n\bar{x}+1)^2}{n(n-1)}$.

With respect to the loss function $L_1$, the Bayes estimator of $\theta$ is $E(\theta | \bar{x}) = \frac{n+1}{n\bar{x} + 1}$.

With respect to the loss function $L_2$, the Bayes estimator of $\theta$ is $E\left( \frac{1}{\theta} | \bar{x} \right) = \frac{n-1}{n\bar{x} + 1}$.

With respect to the loss function $L_3$, the Bayes estimator of $\theta$ is $\{E(\theta^2 | \bar{x})\}^{1/2} = \sqrt{\frac{(n+1)(n+2)}{(n\bar{x} + 1)}}$. 


12. The joint pdf of \( \mathbf{X} = (X_1, X_2, \ldots, X_n) \) is \( f(\mathbf{x} \mid \theta) = \frac{1}{\theta_n}, \ 0 < x_{(1)} < \cdots < x_{(n)} < \theta. \)

The joint pdf of \( \mathbf{X} \) and \( \theta \) is \( f^*(\mathbf{x}, \theta) = \frac{\alpha \beta^\alpha}{\theta^{n+\alpha+1}}, \ \theta > \max \{\beta, x_{(n)}\}. \)

The marginal density of \( \mathbf{X} \) is then \( h(\mathbf{x}) = \frac{\alpha \beta^\alpha}{(n + \alpha) \left( \max \{\beta, x_{(n)}\} \right)^{n+\alpha}}, \ \mathbf{x}_{(n)} > 0. \)

Hence the posterior density of \( \theta \) given \( \mathbf{X} = \mathbf{x} \) is
\[
g^*(\theta \mid \mathbf{x}) = \frac{(n + \alpha) \left( \max \{\beta, x_{(n)}\} \right)^{n+\alpha}}{\theta^{n+\alpha+1}}, \ \theta > \max \{\beta, x_{(n)}\}. \]

With respect to the loss function \( L \), the Bayes estimator of \( \theta \) is
\[
E(\theta \mid \mathbf{X}) = \frac{n + \alpha}{n + \alpha - 1} \max \{\beta, X_{(n)}\}. \]