**Statistical Inference**
**Test Set 3**

1. Let $X_1, X_2, ..., X_n$ be a random sample from a population with density function $f(x) = \frac{x}{\theta} \exp\left\{-\frac{x^2}{2\theta}\right\}, \quad x > 0, \theta > 0.$ Find FRC lower bound for the variance of an unbiased estimator of $\theta$. Hence derive a UMVUE for $\theta$.

2. Let $X_1, X_2, ..., X_n$ be a random sample from a discrete population with mass function
$$P(X = -1) = \frac{1-\theta}{2}, P(X = 0) = \frac{1}{2}, P(X = 1) = \frac{\theta}{2}, \; 0 < \theta < 1.$$
Find FRC lower bound for the variance of an unbiased estimator of $\theta$. Show that the variance of the unbiased estimator $\bar{X} + \frac{1}{2}$ is more than or equal to this bound.

3. Let $X_1, X_2, ..., X_n$ be a random sample from a population with density function $f(x) = \theta(1+x)^{-(1+\theta)}, \quad x > 0, \; \theta > 0.$ Find FRC lower bound for the variance of an unbiased estimator of $1/\theta$. Hence derive a UMVUE for $1/\theta$.

4. Let $X_1, X_2, ..., X_n$ be a random sample from a Pareto population with density $f_X(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}, x > \alpha, \alpha > 0, \beta > 2.$ Find a sufficient statistics when (i) $\alpha$ is known, (ii) when $\beta$ is known and (iii) when both $\alpha, \beta$ are unknown.

5. Let $X_1, X_2, ..., X_n$ be a random sample from a Gamma $(p, \lambda)$ population. Find a sufficient statistics when (i) $p$ is known, (ii) when $\lambda$ is known and (iii) when both $p, \lambda$ are unknown.

6. Let $X_1, X_2, ..., X_n$ be a random sample from a Beta $(\lambda, \mu)$ population. Find a sufficient statistics when (i) $\mu$ is known, (ii) when $\lambda$ is known and (iii) when both $\lambda, \mu$ are unknown.

7. Let $X_1, X_2, ..., X_n$ be a random sample from a continuous population with density function
$f(x) = \frac{\theta}{(1+x)^{1+\theta}}, \quad x > 0, \theta > 0.$ Find a minimal sufficient statistic.

8. Let $X_1, X_2, ..., X_n$ be a random sample from a double exponential population with the density $f(x) = \frac{1}{2} e^{-|x-\theta|}, \; x \in \mathbb{R}, \theta \in \mathbb{R}.$ Find a minimal sufficient statistic.

9. Let $X_1, X_2, ..., X_n$ be a random sample from a discrete uniform population with pmf
$p(x) = \frac{1}{\theta}, x = 1, \ldots, \theta,$ where $\theta$ is a positive integer. Find a minimal sufficient statistic.

10. Let $X_1, X_2, ..., X_n$ be a random sample from a geometric population with pmf $f(x) = (1-p)^{x-1} p$, $x = 1, 2, \dots,$ $0 < p < 1$. Find a minimal sufficient statistic.

11. Let $X$ have a $N(0, \sigma^2)$ distribution. Show that $X$ is not complete, but $X^2$ is complete.

12. Let $X_1, X_2, ..., X_n$ be a random sample from an exponential population with the density $f(x) = \lambda e^{-\lambda x}$, $x > 0, \lambda > 0.$ Show that $Y = \sum_{i=1}^{n} X_i$ is complete.

13. Let $X_1, X_2, ..., X_n$ be a random sample from an exponential population with the density $f(x) = e^{\mu - x}$, $x > \mu, \mu \in \mathbb{R}.$ Show that $Y = X_{(1)}$ is complete.

14. Let $X_1, X_2, ..., X_n$ be a random sample from a $N(\theta, \theta^2)$ population. Show that $(\overline{X}, S^2)$ is minimal sufficient but not complete.

## Hints and Solutions

1. $\log f(x|\theta) = \log x - \log\theta - \dfrac{x^2}{2\theta}$.

$$E\left(\frac{\partial \log f}{\partial\theta}\right)^2 = E\left(\frac{X^2}{2\theta^2} - \frac{1}{\theta}\right)^2 = \frac{E(X^4)}{4\theta^4} - \frac{E(X^2)}{\theta^3} + \frac{1}{\theta^2} = \frac{8\theta^2}{4\theta^4} - \frac{2\theta}{\theta^3} + \frac{1}{\theta^2} = \frac{1}{\theta^2}.$$

So FRC lower bound for the variance of an unbiased estimator of $\theta$ is $\dfrac{\theta^2}{n}$.

Further $T = \dfrac{1}{2n}\sum X_i^2$ is unbiased for $\theta$ and $Var(T) = \dfrac{\theta^2}{n}$. Hence $T$ is UMVUE for $\theta$.

2. $E(X) = \theta - \dfrac{1}{2}$. So $T = \bar{X} + \dfrac{1}{2}$ is unbiased for $\theta$. $Var(T) = \dfrac{1+4\theta-4\theta^2}{4n}$.

$$\frac{\partial}{\partial\theta}\log f(x|\theta) = \begin{cases} (\theta-1)^{-1}, & x=-1 \\ 0, & x=0 \\ \theta^{-1}, & x=1 \end{cases} \quad \text{So we get } E\left(\frac{\partial \log f}{\partial\theta}\right)^2 = \frac{1}{2\theta(1-\theta)}$$

The FRC lower bound for the variance of an unbiased estimator of $\theta$ is $\dfrac{2\theta(1-\theta)}{n}$.

It can be seen that $Var(T) - \dfrac{2\theta(1-\theta)}{n} = \dfrac{(1-2\theta)^2}{4n} \geq 0.$

3. We have $\dfrac{\partial}{\partial\theta}\log f(x|\theta) = \dfrac{1}{\theta} - \log(1+x)$. Further, $E\{\log(1+X)\} = \theta^{-1}$, and $E\{\log(1+X)\}^2 = 2\theta^{-2}$. So $E\left(\dfrac{\partial \log f}{\partial\theta}\right)^2 = \dfrac{1}{\theta^2}$, and FRC lower bound for the variance of an unbiased estimator of $\theta^{-1}$ is $\dfrac{\theta^2}{n}$. Now $T = \dfrac{1}{n}\sum \log(1+X_i)$ is unbiased for $\theta^{-1}$ and $Var(T) = \dfrac{\theta^2}{n}$.

4. Using Factorization Theorem, we get sufficient statistics in each case as below

(i) $\prod_{i=1}^{n} X_i$ ,(ii) $X_{(1)}$ (iii) $\left(X_{(1)}, \prod_{i=1}^{n} X_i\right)$

5. Using Factorization Theorem, we get sufficient statistics in each case as below

(i) $\sum_{i=1}^{n} X_i$ ,(ii) $\prod_{i=1}^{n} X_i$ (iii) $\left(\sum_{i=1}^{n} X_i, \prod_{i=1}^{n} X_i\right)$

6. Using Factorization Theorem, we get sufficient statistics in each case as below

(i) $\prod_{i=1}^{n} X_i$ ,(ii) $\prod_{i=1}^{n}(1-X_i)$ (iii) $\left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n}(1-X_i)\right)$

7. Using Lehmann-Scheffe Theorem, a minimal sufficient statistic is $\prod_{i=1}^{n}(1+X_i)$.

8. Using Lehmann-Scheffe Theorem, a minimal sufficient statistic is the order statistics $(X_{(1)}, \ldots, X_{(n)})$.

9. Using Lehmann-Scheffe Theorem, a minimal sufficient statistic is the largest order statistics $X_{(n)}$.

10. Using Lehmann-Scheffe Theorem, a minimal sufficient statistic is $\sum_{i=1}^{n} X_i$.

11. Since $E(X)=0$ for all $\sigma>0$, but but $P(X=0)=1$ for all $\sigma>0$. Hence $X$ is not complete. Let $W=X^2$. The pdf of $W$ is $f_W(w)=\dfrac{1}{\sigma\sqrt{2\pi w}}e^{-w/2\sigma^2}, w>0$.

Now $E_\sigma g(W)=0$ for all $\sigma>0 \Rightarrow \int_0^\infty g(w)w^{-1/2}e^{-w/2\sigma^2}dw=0$ for all $\sigma>0$. Uniqueness of the Laplace transform implies $g(w)=0$ *a.e.* Hence $X^2$ is complete.

12. Note that $Y=\sum_{i=1}^{n} X_i$ has a Gamma $(n,\lambda)$ distribution. Now proceeding as in Problem 11, it can be proved that $Y$ is complete.

13. Note that the density of $Y=X_{(1)}$ is given by $f_Y(y)=n\,e^{n(\mu-x)}, x>\mu, \mu\in\mathbb{R}.$

$Eg(Y)=0$ for all $\mu\in\mathbb{R}$

$\Rightarrow \int_\mu^\infty n\, g(y)e^{n(\mu-y)}dy=0$ for all $\mu\in\mathbb{R}$

$\Rightarrow \int_\mu^\infty g(y)e^{-ny}dy=0$ for all $\mu\in\mathbb{R}$

Using Lebesgue integration theory we conclude that $g(y)=0$ a.e. Hence $Y$ is complete.

14. Minimal sufficiency can be proved using Lehmann-Scheffe theorem. To see that $(\bar{X}, S^2)$ is not complete, note that $E\left(\dfrac{n}{n+1}\bar{X}^2-S^2\right)=0$ for all $\theta>0$.

However, $P\left(\dfrac{n}{n+1}\bar{X}^2=S^2\right)=0$.