Statistical Inference
Test Set 2

1. Let $X_1, X_2, ..., X_n$ be a random sample from a $U(-\theta, 2\theta)$ population. Find MLE of θ.
2. Let $X_1, X_2, ..., X_n$ be a random sample from a Pareto population with density $f_X(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}, x > \alpha, \alpha > 0, \beta > 2.$ Find the MLEs of α, β.
3. Let $X_1, X_2, ..., X_n$ be a random sample from a $U(-\theta, \theta)$ population. Find the MLE of θ.
4. Let $X_1, X_2, ..., X_n$ be a random sample from a lognormal population with density $f_X(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(\log_e x - \mu)^2\right\}, x > 0.$ Find the MLEs of μ and σ^2.
5. Let $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ be a random sample from a bivariate normal population with parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho$. Find the MLEs of parameters.
6. Let $X_1, X_2, ..., X_n$ be a random sample from an inverse Gaussian distribution with density $f_X(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, x > 0$. Find the MLEs of parameters.
7. Let $(X_1, X_2, ..., X_k)$ have a multinomial distribution with parameters $n = \sum_{i=1}^{k} X_i$, $p_1, \ldots, p_k; 0 \le p_1, \ldots, p_k \le 1, \sum_{j=1}^{k} p_j = 1,$ where n is known. Find the MLEs of $p_1, \ldots, p_k$.
8. Let one observation be taken on a discrete random variable X with pmf $p(x|\theta)$, given below, where $\Theta = \{1, 2, 3\}$ Find the MLE of θ.

		θ		
x		1	2	3
	1	1/2	1/4	1/4
	2	3/5	1/5	1/5
	3	1/3	1/2	1/6
	4	1/6	1/6	2/3

9. Let $X_1, X_2, ..., X_n$ be a random sample from the truncated double exponential distribution with the density
   $f_X(x) = \frac{e^{-|x|}}{2(1-e^{-\theta})}, \ |x| < \theta, \theta > 0.$
   Find the MLE of θ.

10. Let $X_1, X_2, ..., X_n$ be a random sample from the Weibull distribution with the density
    $f_X(x) = \alpha\, \beta x^{\beta-1} e^{-\alpha x^\beta}, \ x > 0, \alpha > 0, \beta > 0.$
    Find MLE of α when β is known.

Hints and Solutions

1. The likelihood function is $L(\theta,\underline{x})=\frac{1}{(3\theta)^n}, -\theta < x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)} < 2\theta, \theta>0.$ Clearly it is maximized with respect to θ, when θ takes its infimum. Hence, $\hat{\theta}_{ML}=\max\left(-X_{(1)}, \frac{X_{(n)}}{2}\right)$.

2. The likelihood function is $L(\alpha,\beta,\underline{x})=\frac{\beta^n \alpha^{n\beta}}{(\prod_{i=1}^{n} x_i)^{\beta+1}}, x_{(1)}>\alpha, \alpha>0, \beta>2.$ L is maximized with respect to α when α takes its maximum. Hence $\hat{\alpha}_{ML}=X_{(1)}$. Using this we can rewrite the likelihood function as $L'(\beta,\underline{x})=\frac{\beta^n \{x_{(1)}\}^{n\beta}}{(\prod_{i=1}^{n} x_i)^{\beta+1}}, \beta>2.$ The log likelihood is $\log L'(\beta,\underline{x})=n\log\beta+n\beta\log x_{(1)}-(\beta+1)\log\left(\prod_{i=1}^{n} x_i\right)$. This can be easily maximized with respect to β and we get $\hat{\beta}_{ML}=\left[\frac{1}{n}\sum \log\frac{X_{(i)}}{X_{(1)}}\right]^{-1}$.

3. Arguing as in Sol. 1, we get $\hat{\theta}_{ML}=\max\left(-X_{(1)}, X_{(n)}\right)=\max_{1\le i\le n}|X_i|$.

4. Directly maximizing the log-likelihood function with respect to μ and σ^2, we get $\hat{\mu}_{ML}=\frac{1}{n}\sum\log X_i, \hat{\sigma}^2_{ML}=\frac{1}{n}\sum(\log X_i-\hat{\mu}_{ML})^2$.

5. The maximum likelihood estimators are given by
$$\hat{\mu}_1=\overline{X}, \hat{\mu}_2=\overline{Y}, \hat{\sigma}_1^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2, \hat{\sigma}_2^2=\frac{1}{n}\sum_{i=1}^{n}(Y_i-\overline{Y})^2, \hat{\rho}=\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})(Y_i-\overline{Y})/(\hat{\sigma}_1\hat{\sigma}_2).$$

6. The maximum likelihood estimators are given by
$$\hat{\mu}_{ML}=\overline{X}, \hat{\lambda}_{ML}=\left[\frac{1}{n}\left(\sum_{i=1}^{n}\frac{1}{X_i}-\frac{1}{\overline{X}}\right)\right]^{-1}$$

7. The maximum likelihood estimators are given by
$$\hat{p}_1=\frac{X_1}{n},\cdots,\hat{p}_k=\frac{X_k}{n}$$

8. $\hat{\theta}_{ML}=1,\ \text{if } x=1,2$
   $=2,\ \text{if } x=3$
   $=3,\ \text{if } x=4$

9. $\hat{\theta}_{ML}=\max\left(-X_{(1)}, X_{(n)}\right)=\max_{1\le i\le n}|X_i|$.

10. $\hat{\alpha}_{ML}=\frac{n}{\sum x_i^{\beta}}$