1. Let $X \sim P(\lambda)$. Find unbiased estimators of $(i) \lambda^3$, $(ii) e^{-\lambda}$, $(iii) \cos \lambda$. $(iv)$ Show that there does not exist unbiased estimators of $1/\lambda$, and $\exp\{-1/\lambda\}$.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ population. Find unbiased and consistent estimators of the signal to noise ration $\frac{\mu}{\sigma}$ and quantile $\mu + b\sigma$, where $b$ is any given real.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $U(-\theta, 2\theta)$ population. Find an unbiased and consistent estimator of $\theta$.

4. Let $X_1, X_2$ be a random sample from an exponential population with mean $1/\lambda$. Let $T_1 = \frac{X_1 + X_2}{2}, T_2 = \sqrt{X_1X_2}$. Show that $T_1$ is unbiased and $T_2$ is biased. Further, prove that $\text{MSE}(T_1) \leq \text{Var}(T_1)$.

5. Let $T_1$ and $T_2$ be unbiased estimators of $\theta$ with respective variances $\sigma_1^2$ and $\sigma_2^2$ and $\text{cov}(T_1, T_2) = \sigma_{12}$ (assumed to be known). Consider $T = \alpha T_1 + (1-\alpha)T_2, 0 \leq \alpha \leq 1$. Show that $T$ is unbiased and find value of $\alpha$ for which $\text{Var}(T)$ is minimized.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from an $\text{Exp}(\mu, \sigma)$ population. Find the method of moment estimators (MMEs) of $\mu$ and $\sigma$.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Pareto population with density $f_X(x) = \frac{\beta \alpha^\beta}{x^{\beta+1}}, x > \alpha, \alpha > 0, \beta > 2$. Find the method of moments estimators of $\alpha, \beta$.

8. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $U(-\theta, \theta)$ population. Find the MME of $\theta$.

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from a lognormal population with density $f_X(x) = \frac{1}{\sigma x \sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2} (\log x - \mu)^2\right\}, x > 0$. Find the MMEs of $\mu$ and $\sigma^2$.

10. Let $X_1, X_2, \ldots, X_n$ be a random sample from a double exponential $(\mu, \sigma)$ population. Find the MMEs of $\mu$ and $\sigma$. 

Statistical Inference  
Test Set 1
Hints and Solutions

1. (i) \( E \{X(X - 1)(X - 2)\} = \lambda^3 \)

(ii) For this we solve estimating equation. Let \( T(X) \) be unbiased for \( e^{-\lambda} \cos \lambda \).

Then \( ET(X) = e^{-\lambda} \cos \lambda \) for all \( \lambda > 0 \).

\[
\Rightarrow \sum_{x=0}^{\infty} T(x) \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \cos \lambda \text{ for all } \lambda > 0
\]

\[
\Rightarrow \sum_{x=0}^{\infty} T(x) \frac{\lambda^x}{x!} = 1 - \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} - \cdots \text{ for all } \lambda > 0
\]

As the two power series are identical on an open interval, equating coefficients of powers of \( \lambda \) on both sides gives

\( T(x) = 0 \), if \( x = 2m + 1 \),

\( = 1 \), if \( x = 4m \),

\( = -1 \), if \( x = 4m + 2 \), \( m = 0, 1, 2, \ldots \)

(iii) For this we have to solve estimating equation. However, we use Euler’s identity to solve it.

Let \( U(X) \) be unbiased for \( \sin \lambda \). Then

\[
\sum_{x=0}^{\infty} U(x) \frac{\lambda^x}{x!} = \frac{1}{2i} e^{i\lambda} (e^{i\lambda} - e^{-i\lambda}) \text{ for all } \lambda > 0
\]

\[
= \frac{1}{2i} (e^{(1+i)\lambda} - e^{-(1-i)\lambda}) \text{ for all } \lambda > 0
\]

\[
= \frac{1}{2i} \left( \sum_{k=0}^{\infty} \frac{\lambda^k (1+i)^k}{k!} - \sum_{k=0}^{\infty} \frac{\lambda^k (1-i)^k}{k!} \right) \text{ for all } \lambda > 0.
\]

Applying De-Moivre’s Theorem on the two terms inside the parentheses, we get

\[
\sum_{x=0}^{\infty} U(x) \frac{\lambda^x}{x!} = \frac{1}{2i} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left[ (\sqrt{2})^k \left( \cos \frac{\pi x}{4} + i \sin \frac{\pi x}{4} \right)^k - (\sqrt{2})^k \left( \cos \left( -\frac{\pi x}{4} \right) + i \sin \left( -\frac{\pi x}{4} \right) \right)^k \right]
\]

\[
= \frac{1}{2i} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left[ (\sqrt{2})^k \left( \cos \frac{k\pi x}{4} + i \sin \frac{k\pi x}{4} \right) - (\sqrt{2})^k \left( \cos \left( -\frac{k\pi x}{4} \right) + i \sin \left( -\frac{k\pi x}{4} \right) \right) \right]
\]

\[
= \sum_{k=0}^{\infty} \frac{(\sqrt{2})^k \lambda^k}{k!} \sin \left( \frac{k\pi x}{4} \right) \text{ for all } \lambda > 0
\]

Equating the coefficients of powers of \( \lambda \) on both sides gives

\( U(x) = (\sqrt{2})^x \sin \left( \frac{\pi x}{4} \right), x = 0, 1, 2, \ldots \)

In Parts (iv) and (v), we can show in a similar way that estimating equations do not have any solutions.
2. Let \( \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \), and \( S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \).

Then \( \bar{X} \sim N(\mu, \sigma^2/n) \), and \( W = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \). It can be seen that \[ E(W^{1/2}) = \frac{\sqrt{2}}{2} \frac{n-1}{n-2} \quad \text{and} \quad E(W^{-1/2}) = \frac{\sqrt{2}}{2} \frac{n-1}{n-2} \]. Using these, we get unbiased estimators of \( \sigma \) and \( \frac{1}{\sigma} \) as \( T_1 = \frac{\sqrt{n-1}}{n-2} \frac{S}{\bar{X}} \) and \( T_2 = \frac{\sqrt{2}}{n-1} \frac{\sqrt{n-2}}{S} \) respectively. As \( \bar{X} \) and \( S^2 \) are independently distributed, \( U_1 = \frac{\sqrt{n-1}}{n-2} \frac{S}{\bar{X}} \) is unbiased for \( \frac{\mu}{\sigma} \). Further, \( U_2 = \bar{X} + bT_1 \) is unbiased for \( \mu + b\sigma \). As \( \bar{X} \) and \( S^2 \) are consistent for \( \mu \) and \( \sigma^2 \) respectively, \( U_1 \) and \( U_2 \) are also consistent for \( \frac{\mu}{\sigma} \) and \( \mu + b\sigma \) respectively.

3. As \( \mu_i = \frac{3\theta}{2}, T = \frac{2\bar{X}}{3} \) is unbiased for \( \theta \). \( T \) is also consistent for \( \theta \).

4. As \( E(X_i) = \frac{1}{\lambda}, T_i \) is unbiased. Also \( X_1 \) and \( X_2 \) are independent. So

\[
E(T_2) = E\left( \frac{1}{\sqrt{\lambda}} \right) = \left( E\left( \frac{1}{\sqrt{\lambda}} \right) \right)^2 = \left( \frac{1}{2} \sqrt{\pi/\lambda} \right)^2 = \frac{\pi}{4\lambda}, \quad \text{Var}(T_i) = \frac{1}{2\lambda^2}.
\]

\[
MS \; KT_2 = E\left( \frac{1}{\sqrt{\lambda}} \right)^2 = E(X_i^2) - \frac{2}{\lambda} E\left( \frac{1}{\sqrt{\lambda}} \right) + \frac{1}{\lambda^2}
\]

\[
= \frac{2}{\lambda^2} \left( 1 - \frac{\pi}{4} \right)
\]

5. The minimizing choice of \( \alpha \) is obtained as \( \frac{\sigma^2 - \sigma_{12}}{\sigma_i^2 + \sigma_{12}^2 - 2\sigma_{12}} \).

6. \( f(x) = \frac{1}{\sigma} \exp\left( -\frac{x-\mu}{\sigma} \right), x > \mu, \sigma > 0. \) \( \mu' = \mu + \sigma, \mu'_2 = (\mu + \sigma)^2 + \sigma^2 \).

So \( \mu = \frac{1}{2}(\mu' - \mu'^2 - \mu'^2), \sigma = \sqrt{\mu'_2 - \mu'^2} \). The method of moments estimators for \( \mu \) and \( \sigma \) are therefore given by

\[
\hat{\mu}_{MM} = \bar{X} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \bar{X})^2, \quad \hat{\sigma}_{MM} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2}.
\]
7. \( \mu'_1 = \frac{\beta \alpha}{\beta - 1}, \mu'_2 = \frac{\beta \alpha^2}{\beta - 2} \). So \( \alpha = -\frac{\mu'_1}{\sqrt{\mu'_2 - \mu'_1^2}}, \beta = 1 + \sqrt{\frac{\mu'_2}{\mu'_2 - \mu'_1^2}} \).

The method of moments estimators for \( \alpha \) and \( \beta \) are therefore given by

\[
\hat{\alpha}_{MM} = \frac{\bar{X} \sqrt{\sum_{i=1}^{n} X_i^2}}{\sqrt{\sum_{i=1}^{n} (X_i - \bar{X})^2 + \sum_{i=1}^{n} X_i^2}}, \quad \hat{\beta}_{MM} = 1 + \sqrt{\frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n} (X_i - \bar{X})^2}}.
\]

8. Since \( \mu'_1 = 0 \), we consider \( \mu'_2 = \frac{\theta^2}{3} \). So \( \hat{\theta}_{MM} = \sqrt{\frac{3}{n} \sum_{i=1}^{n} X_i^2} \).

9. \( \mu'_1 = e^{\mu + \sigma^2/2}, \mu'_2 = e^{2\mu + 2\sigma^2} \). So \( \mu = \log \left( \frac{\mu'_2}{\sqrt{\mu'_1}} \right), \sigma^2 = \log \left( \frac{\mu'_2}{\mu'_1} \right) \) and the method of moments estimators for \( \mu \) and \( \sigma^2 \) are therefore given by

\[
\hat{\mu}_{MM} = \log \left( \frac{\bar{X}^2}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} X_i^2}} \right), \quad \hat{\sigma}_{MM}^2 = \log \left( \frac{1}{n} \sum_{i=1}^{n} X_i^2 \right).
\]

10. \( f(x) = \frac{1}{2\sigma} \exp \left( -\frac{|x - \mu|}{\sigma} \right), x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0. \mu'_1 = \mu, \mu'_2 = \mu^2 + 2\sigma^2 \).

So \( \mu = \mu'_1, \sigma = \sqrt{\frac{1}{2} (\mu'_2 - \mu'_1^2)} \). The method of moments estimators for \( \mu \) and \( \sigma \) are therefore given by

\[
\hat{\mu}_{MM} = \bar{X}, \quad \hat{\sigma}_{MM} = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} (X_i - \bar{X})^2}.
\]