Analysis of Variance and Design of Experiments-II

MODULE - III

LECTURE - 16

PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN (PBIBD)

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In order to learn how to write these matrices $P_1$ and $P_2$, we consider the treatments 1, 2 and 8. Note that the treatment 8 is the second associate of treatment 1. Consider only the rows corresponding to treatments 1, 2 and 8 and obtain the elements of $P_1$ and $P_2$ as follows:

$p_{11}^1$: Treatments 1 and 2 are the first associates of each other. There are three common treatments (viz., 3, 4 and 5) between the first associates of treatment 1 and the first associates of treatment 2. So $p_{11}^1 = 3$. 

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{array}
\]
$p_{12}^1$ and $p_{21}^1$: Treatments 1 and 2 are the first associates of each other. There are two treatments (viz., 6 and 7) which are common between the first associates of treatment 1 and the second associates of treatment 2. So $p_{12}^1 = 2 = p_{21}^1$. 

1. **Treatments 1 and 2 are the first associates of each other.**
2. **There are two treatments (viz., 6 and 7) which are common between the first associates of treatment 1 and the second associates of treatment 2.**
3. **So $p_{12}^1 = 2 = p_{21}^1$.**
$p_{22}^1$ : Treatments 1 and 2 are the first associates of each other. There is only one treatment (viz., treatment 10) which is common between the second associates of treatment 1 and the second associates of treatment 2. So $p_{22}^1 = 1$. 

![Diagram showing the associates of treatments 1 and 2]
$p_{11}^2$: Treatments 1 and 8 are the second associates of each other. There are four treatments (viz., 2, 3, 5, and 6) which are common between the first associates of treatment 1 and the first associates of treatment 8. So $p_{11}^1 = 4$. 

Those four treatments which are the second associates of treatments 1 and 2.
$p_{12}^2$ and $p_{21}^2$: Treatments 1 and 8 are the second associates of each other. There are two treatments (viz., 4 and 7) which are common between the first associates of treatment 1 and the second associates of treatment 8. So $p_{12}^1 = 2 = p_{21}^1$. 

Those two treatments which are the 1st associate of treatments 1 and 8.
$p_{22}^2$: Treatments 1 and 8 are the second associates of each other. There is no treatment which is common between the second associates of treatment 1 and the second associates of treatment 8. So $p_{22}^2 = 0$. 

There are no treatments which are the 2nd associates of treatments 1 and 8.
In general, if \( q \) rows and \( q \) columns of a square are used, then for \( q > 3 \)

\[
v = \binom{q}{2} = \frac{q(q-1)}{2},
\]

\[
n_1 = 2q - 4,
\]

\[
n_2 = \frac{(q-2)(q-3)}{2},
\]

\[
P_1 = \begin{bmatrix}
q-2 & q-3 \\
q-3 & \frac{(q-3)(q-4)}{2}
\end{bmatrix}
\]

\[
P_1 = \begin{bmatrix}
4 & 2q-8 \\
2q-8 & \frac{(q-4)(q-5)}{2}
\end{bmatrix}.
\]

For \( q = 3 \), there are no second associates. This is a degenerate case where second associates do not exist and hence \( P_2 \) cannot be defined.

The graph theory techniques can be used for counting \( p_{jk}^i \).

Further, it is easy to see that all the parameters in \( P_1, P_2 \) etc. are not independent.
Construction of blocks of PBIBD under triangular association scheme

The blocks of a PBIBD can be obtained in different ways through an association scheme. Even different block structure can be obtained using the same association scheme. We now illustrate this by obtaining different block structures using the same triangular scheme. Consider the earlier illustration where $v = \binom{q}{2} = 10$ with $q = 5$ was considered and the first, second and third associates of treatments were obtained.

**Approach 1** One way to obtain the treatments in a block is to consider the treatments in each row. This constitutes the set of treatments to be assigned in a block. When $q = 5$, the blocks of PBIBD are constructed by considering the rows of the following table.

<table>
<thead>
<tr>
<th>Rows</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>×</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>×</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>5</td>
<td>×</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>×</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>10</td>
<td>×</td>
</tr>
</tbody>
</table>
From this arrangement, the treatments are assigned in different blocks and following blocks are obtained.

<table>
<thead>
<tr>
<th>Blocks</th>
<th>Treatments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block 1</td>
<td>1, 2, 3, 4</td>
</tr>
<tr>
<td>Block 2</td>
<td>1, 5, 6, 7</td>
</tr>
<tr>
<td>Block 3</td>
<td>2, 5, 8, 9</td>
</tr>
<tr>
<td>Block 4</td>
<td>3, 6, 8, 10</td>
</tr>
<tr>
<td>Block 5</td>
<td>4, 7, 9, 10</td>
</tr>
</tbody>
</table>

The parameters of such a design are \( b = 5, v = 10, r = 2, k = 4, \lambda_1 = 1 \) and \( \lambda_2 = 0 \).
Consider e.g., the triangular association scheme for $q = 5$. Then the first block under this approach is obtained by deleting the common treatments between columns 1 and 2 which results in a block containing the treatments 2, 3, 4, 5, 6 and 7.

Similarly, considering the other pairs of columns, the other blocks can be obtained which are presented in the following table:

<table>
<thead>
<tr>
<th>Blocks</th>
<th>Columns of association scheme</th>
<th>Treatments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block 1</td>
<td>(1, 2)</td>
<td>2, 3, 4, 5, 6, 7</td>
</tr>
<tr>
<td>Block 2</td>
<td>(1, 3)</td>
<td>1, 3, 4, 5, 8, 9</td>
</tr>
<tr>
<td>Block 3</td>
<td>(1, 4)</td>
<td>1, 2, 4, 6, 8, 10</td>
</tr>
<tr>
<td>Block 4</td>
<td>(1, 5)</td>
<td>1, 2, 3, 7, 9, 10</td>
</tr>
<tr>
<td>Block 5</td>
<td>(2, 3)</td>
<td>1, 2, 6, 7, 8, 9</td>
</tr>
<tr>
<td>Block 6</td>
<td>(2, 4)</td>
<td>1, 3, 5, 7, 8, 10</td>
</tr>
<tr>
<td>Block 7</td>
<td>(2, 5)</td>
<td>1, 4, 5, 6, 9, 10</td>
</tr>
<tr>
<td>Block 8</td>
<td>(3, 4)</td>
<td>2, 3, 5, 6, 9, 10</td>
</tr>
<tr>
<td>Block 9</td>
<td>(3, 5)</td>
<td>2, 4, 5, 7, 8, 10</td>
</tr>
<tr>
<td>Block 10</td>
<td>(4, 5)</td>
<td>3, 4, 6, 7, 8, 9</td>
</tr>
</tbody>
</table>

The parameters of the PBIBD are $b = 10$, $v = 10$, $r = 6$, $k = 6$, $\lambda_1 = 3$ and $\lambda_2 = 4$. 

**Approach 2**

Another approach to obtain the blocks of PBIBD from a triangular association scheme is as follows:

- Consider the pairwise columns of the triangular scheme.
- Then delete the common treatments between the chosen columns and retain others.
- The retained treatments will constitute the blocks.
Since both these PBIBDs are arising from same association scheme, so the values of \( n_1, n_2, P_1 \) and \( P_2 \) remain the same for both the designs. In this case, we have

\[
\begin{align*}
n_1 &= 6, \quad n_2 = 3, \\
P_1 &= \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}.
\end{align*}
\]

**Approach 3** Another approach to derive the blocks of PBIBD is to consider all the first associates of a given treatment in a block. For example, in case of \( q = 5 \), the first associates of treatment 1 are the treatments 2, 3, 4, 5, 6 and 7. So these treatments constitute one block. Similarly other blocks can also be found. This results in the same arrangement of treatments in blocks as considered by deleting the common treatments between the pair of columns.

The PBIBD with two associate classes are popular in practical applications and can be classified into following types depending on the association scheme,


1. Triangular
2. Group divisible
3. Latin square with \( i \) constraints
4. Cyclic and
5. Singly linked blocks

The triangular association scheme has already been discussed. We now briefly present other types of association schemes.