Module 3

Problem Sheet

1. Show that an i.i.d sequence of continuous random variable with common probability density function \( f \) is strictly stationary.

2. Find (under certain conditions) whether the stochastic process \( \{X(t), t \in T\} \) with probability distribution given by:
   \[
P(X(t) = n) = \begin{cases} 
   \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, \ldots \\
   \frac{at}{1+at}, & n = 0
   \end{cases}
   \]
is stationary.

3. Let \( X(t) = A_0 + A_1 t + A_2 t^2 \) where \( A_i \)'s are uncorrelated random variables with mean 0 and variance 1. Find the mean function and covariance function of \( X(t) \).

4. Let \( Y_n = a_0 X_n + a_1 X_{n-1}, n = 1, 2, \ldots \) where \( a_0, a_1 \) are constants and \( X_0, X_1, \ldots \), are i.i.d. random variables with mean 0 and variance \( \sigma^2 \).
   (a) Is \( \{Y_n, n \geq 1\} \) covariance stationary?

5. Consider autoregressive process of order 1, i.e.
   \[X_t = c + \phi X_{t-1} + \varepsilon_t\]
   where \( \varepsilon_t \) is white noise with mean 0 and variance \( \sigma^2 \), \( c \) is a constant. Assume that the mean of the random variable \( X_t \) is identical for all values of \( t \), denoted by \( \mu \). Show that the process is wide sense stationary for \( |\phi| < 1 \).

6. Let \( \{N(t), t \geq 0\} \) be a Poisson Process. Prove or disprove that \( \{X(t) = N(t+L) - N(t), t \geq 0\} \), where \( L \) is a positive constant, is covariance or wide-sense stationary.

7. Let \( Z_1 \) and \( Z_2 \) be two independent normal random variables with mean 0 and variance \( \sigma^2 \). Define
   \[X(t) = Z_1 \cos(\lambda t) + Z_2 \sin(\lambda t)\]
   Then show that \( \{X(t), t \in T\} \) is a second order stationary process.

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Answers to Problem Sheet

Ans 1. Let $X_1, X_2, \ldots$, be an i.i.d. sequence of continuous random variables.

Let $n$ be any positive integer.

Let $m \in \mathbb{Z}$ such that $n + m > 0$.

Then $P(X_{1+m}, X_{2+m}, \ldots, X_{n+m}) \in B$ and its distribution is:

$$\int \cdots \int_B f(x_{1+m})f(x_{2+m})\cdots f(x_{n+m})dx_{1+m}dx_{2+m}\cdots dx_{n+m}$$

Since $X_i$’s are i.i.d. random variables and $x_{1+m}, x_{2+m}, \ldots x_{n+m}$ are just dummy variables of integration, we may replace them by $x_1, x_2, \ldots, x_n$.

Hence above integral is equal to

$$\int \cdots \int_B f(x_1)f(x_2)\cdots f(x_n)dx_1dx_2\cdots dx_n$$

which is independent of $m$ and hence the process is strictly stationary.

Ans 2. Given $P[X(t) = n] = \left\{ \begin{array}{ll} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, \ldots \\ \frac{at}{1+at}, & n = 0 \end{array} \right.$

(i) $E[X(t)] = \sum_{n=0}^{\infty} nP(X(t) = n) = \sum_{n=1}^{\infty} \frac{n(at)^{n-1}}{(1+at)^{n+1}} = \frac{1}{(1+at)^2}(1+at)^2 = 1$

(ii) $E[X^2(t)] = \sum_{n=1}^{\infty} n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}} = \frac{1}{(1+at)^2}\sum_{n=1}^{\infty} n^2 \left( \frac{at}{1+at} \right)^{n-1}$

Ans 3. Let $X(t) = A_0 + A_1 t + A_2 t^2$ where

$E(A_i) = 0 \forall i$, $\text{Var}(A_i) = 1 \forall i$ and $\text{Cov}(A_i, A_j) = 0 \forall i \neq j$.

(a) Mean function of $X(t)$:

$E[X(t)] = E[A_0 + A_1 t + A_2 t^2] = E[A_0] + t E[A_1] + t^2 E[A_2] = 0$

(b) Covariance function of $X(t)$:

$\text{Cov}(X(t_1), X(t_2)) = E[X(t_1)X(t_2)] - E[X(t_1)]E[X(t_2)]$

$= E[X(t_1)X(t_2)]$

$= E[(A_0 + A_1 t_1 + A_2 t_1^2)(A_0 + A_1 t_2 + A_2 t_2^2)]$

$= E[A_0^2 + A_0 A_1 t_2 + A_0 A_2 t_2^2 + A_1 A_0 t_1 + A_1^2 t_1 t_2 + A_1 A_2 t_2^2 + A_2 A_0 t_1^2 + A_1 A_2 t_1^2 t_2 + A_2 A_2 t_1^2 t_2]$

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Now, as Cor(A_i, A_j) = 0 ∀ i ≠ j, therefore:


Hence

\[
\text{Cov}(X(t_1), X(t_2)) = E[A_1^2] + t_1E[A_0]E[A_1] + t_2E[A_0]E[A_2] + t_1t_2E[A_1]E[A_2] + t_1^2E[A_0]E[A_1] + t_2^2E[A_0]E[A_2] + t_1t_2E[A_1]E[A_2] + t_1^2t_2E[A_2] + t_1^2t_2E[A_2] = 1 + t_1t_2 + t_1^2t_2 \] (\( \because E[A_i] = 0 \) ∀i).

**Ans 4.** \( Y_n = a_0X_n + a_1X_{n-1} \), \( n = 1, 2, \ldots \) where \( a_i \)'s are constants and \( X_0, X_1, \ldots, \) are i.i.d's random variables with \( E(X_i) = 0 \) and \( \text{Var}X_i = \sigma^2 \).

(a) Is \( Y_n \) covariance stationary:

(i) \( E[Y_n] = E[a_0X_n + a_1X_{n-1}] = 0 \)

(ii) \( E[Y_n^2] = E[(a_0X_n + a_1X_{n-1})^2] \)
\[
= E[a_0^2X_n^2 + a_1^2X_{n-1}^2 + 2a_0a_1X_nX_{n-1}] = a_0^2\sigma^2 + a_1^2\sigma^2 + 2a_0a_1E(X_nX_{n-1}) = a_0^2\sigma^2 + a_1^2\sigma^2 + a_0a_1(E(X_n)E(X_{n-1})) \] (\( \because \) they are i.i.d)
\[
= a_0^2\sigma^2 + a_1^2\sigma^2 (\because E(X_i) = 0)
\]

(iii) \( \text{Cov}(Y_n, Y_m) = \text{Cov}(a_0X_n + a_1X_{n-1}, a_0X_m + a_1X_{m-1}) \)
\[
= E[(a_0X_n + a_1X_{n-1})(a_0X_m + a_1X_{m-1})] (\because E(Y_n) = E(Y_m) = 0) = E[a_0^2X_nX_m + a_0a_1X_nX_{m-1} + a_1a_0X_mX_{n-1} + a_1^2X_{m-1}X_{n-1}] \]
\[
= \begin{cases} 
  a_0^2\sigma^2 + a_1^2\sigma^2, & n=m; \\
  a_0a_1\sigma^2, & n=m-1; \\
  a_0a_1\sigma^2, & n=m+1; \\
  0, & \text{otherwise.}
\end{cases}
\]

which is a function of \( n - m \).

Hence \( Y_n \) is covariance stationary.

**Ans 5. (i)** First calculating expectation
\[ E(X_t) = E(c + \phi X_{t-1} + \varepsilon_t) \]
\[ \mu = c + \phi \mu + 0 \]
\[ \Rightarrow \mu = \frac{c}{1-\phi} \]
which is independent of \( t \).

\[ \text{Var}(X_t) = \sigma^2_X \text{ and } \] (1)

\[ \text{Var}(X_t) = \text{Var}[c + \phi X_{t-1} + \varepsilon_t] \]
\[ = \phi^2 \text{Var}(X_{t-1}) + \sigma^2 \varepsilon \] (2)

Since \( \{X_t : t \in T\} \) are identical, \( \Rightarrow \text{Var}(X_t) = \text{Var}(X_{t-1}) \)

Equating (1) and (2):
\[ \sigma^2_X = \phi^2 \sigma^2_X + \sigma^2 \varepsilon \]
\[ \sigma^2_X = \frac{\sigma^2}{1-\phi^2} \Rightarrow \text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2} \]
which exists and is finite for \( |\phi| < 1 \).

(iii) Since \( X_t \)'s are identical
\[ E(X_{t_1}, X_{t_2}) = \mu^2 \text{ and } \]
\[ \text{Cov}(X_{t_1}, X_{t_2}) = 0 \]
which are functions of \( |t_1 - t_2| \).

Hence the process is wide sense stationary.

Ans 6. We have \( X(t) = N(t+L) - N(t) \sim P(\lambda(t+L-t)) = P(\lambda L) \)

(a) \( E(X(t)) = \lambda L \) which is independent of \( t \).

(b) \( E(X^2(t)) = \lambda L + (\lambda L)^2 < \infty \) \( \forall t \).

(c) Let \( s < t \).

\[ \text{cov}(X(t), X(s)) = E(X(t)X(s)) - E(X(t))E(X(s)) \]
\[ = E((X(t) - X(s) + X(s))X(s)) - (\lambda L)^2 \]
\[ = E(X(t) - X(s))E(X(s)) + E(X^2(s)) - (\lambda L)^2 \]
\[ = 0 \ast E(X(s)) + \lambda L \]
\[ = \lambda L \]

which is constant function. So we can consider it as a function of \( t - s \).

From (a),(b) and (c) \{\( X(t), \ t \geq 0 \}\} is covariance stationary.
Ans 7. (a) $E(X(t)) = E(Z_1)\cos(\lambda t) + E(Z_2)\sin(\lambda t)$

$= 0$ which is independent of $t$.

(b) $E(X^2(t)) = \cos^2(\lambda t)E(Z_1^2) + \sin^2(\lambda t)E(Z_2^2) + 2\cos(\lambda t)\sin(\lambda t)E(Z_1)E(Z_2)$

$= \cos^2(\lambda t)\sigma^2 + \sin^2(\lambda t)\sigma^2 + 2\cos(\lambda t)\sin(\lambda t) \ast 0$

$= \sigma^2 < \forall t$

From (a),(b) $\{X(t), \ t \geq 0\}$ is second order stationary.