1. Obtain an orthonormal basis for $v$, the space of all real polynomials of degree at most 2, the inner product being defined by,

$$(f, g) = \int_{0}^{1} f(x)g(x)dx$$

...(1)

Solution. We have $v = \{a_0 + a_1x + a_2x^2 : a_i \in R\}$

Clearly $\{v_1 = 1, v_2 = x, v_3 = x^2\}$ is a basis of $V$.

Let $w_1 = v_1$ so that $||w_1||^2 = (w_1|w_1) = (1|1)$ or

Using (1)

$$||w_1||^2 = \int_{0}^{1} 1.1dx$$

∴ $w_1 = 1$ let $w_2 = v_2 - (v_2|w_1)w_1$

(2)

We have, $v_3 = (v_2, v_1) = \int_{0}^{1} x.1dx = \frac{1}{2}$ So by (2) we get,

$w_2 = x - \frac{1}{2}$

$\Rightarrow ||w_2||^2 = \int_{0}^{1} w_2w_2dx$

$$= \int_{0}^{1} (x - \frac{1}{2})^2 dx$$

$$= \frac{1}{12}$$

∴ $\frac{w_2}{||w_2||} = \sqrt{12}\left(x - \frac{1}{2}\right)$

Let $w_3 = v_3 - (v_3|w_1)w_1 - (v_3|w_2)w_2$

We have, $v_3|w_1 = \int_{0}^{1} v_3w_1dx$

$$= \int_{0}^{1} x^2.1dx$$

$$= \frac{1}{3}$$

$v_3|w_2 = \int_{0}^{1} v_3w_2dx$

$$= \int_{0}^{1} x^2(x - \frac{1}{2})dx$$

$$= \frac{1}{4} - \frac{1}{6}$$

$$= \frac{1}{12}$$

and $||w_2|| = 1$

$||w_2||^2 = \frac{1}{12}$
putting in (3), we get,

\[ w_3 = x^2 - \frac{1}{3} - 1 - \left(x - \frac{1}{2}\right) dx \]
\[ = x^2 - x + \frac{1}{6} \]

\[ ||w_3||^2 = (w_3|w_3) \]
\[ = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx \]
\[ = \frac{1}{180} \]
\[ \therefore \frac{w_3}{||w_3||} = \sqrt{\frac{180}{x^2 - x + \frac{1}{6}}} \]

Hence an orthonormal basis for \( V \) is, \( \left\{ 1, 2 + \sqrt{3} \left(x - \frac{1}{2}\right), 6\sqrt{5} \left(x^2 - x + \frac{1}{6}\right) \right\} \)

2. Show that in a complex inner product space \( v \). If \( x \) is orthogonal to \( y \) then \( ||x + y||^2 = ||x||^2 + ||y||^2 \).

However the converse need not be true. Justify.

**Solution.** We have, \((x|y) = 0 \Rightarrow (y|x) = (x,y) = 0 = 0 \)

\[ \therefore ||x + y||^2 = (x + y, x + y) \]
\[ = (x|x) + (x|y) + (y|x) + (y|y) \]
\[ = ||x||^2 + ||y||^2 \]

However the converse need not be true.

Consider, \( V = C^2 \) with standard inner product.

Let \( x = (0, i) \) \( y = (0, 1) \in V \) then

\((x|y) = 0.0 + i.1 = i \neq 0 \Rightarrow x \) is not orthogonal to \( y \).

Now \( ||x||^2 = 0.0 + i.7 = i(-i) = 1 \)
\( ||y||^2 = 0.1 + 1.1 = 1 \) We have, \((0, (1 + i)), \) and so \( ||x + y||^2 = 0.0 + (1 + i)(1 + i) \)

or \( ||x + y||^2 = (1 + i)(1 - i) = 1 - i^2 = 2 \)

\[ \Rightarrow ||x + y||^2 = ||x||^2 + ||y||^2, \] but \( x \) is not orthogonal to \( y \).

3. A \( 2 \times 2 \) real symmetric matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is positive definite iff the diagonal entries \( a \) and \( d \) are positive and the determinant \( |A| = ad - bc = ad - b^2 \) is positive.

**Solution.** To prove \( A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \) is positive definite iff \( a \) and \( d \) are positive and \( |A| = ad - b^2 \) is positive.

Let \( u = [x, y]^T \), then
\[ f(u) = u^T A u \]
\[ = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]
\[ = ax^2 + 2bxy + dy^2 \]

Suppose \( f(u) > 0 \) for every \( u \neq 0 \), then \( f(1, 0) = a > 0 \) and \( f(0, 1) = d > 0 \).

Also we have \( f(b, -a) = a(ad - b^2) > 0 \) since \( a > 0 \), we get \( ad - b^2 > 0 \). Conversely suppose \( a > 0, b = 0, ad - b^2 > 0 \). Completing the square give us,

\[ f(u) = a(x^2 + \frac{2b}{a}xy + \frac{b^2}{a}y^2) + dy^2 - \frac{b^2}{a}y^2 \]
\[ = a \left( x + \frac{by}{a} \right)^2 + \frac{ad - b^2}{a} y^2 \]

Accordingly \( f(u) > 0 \) for every \( u \neq 0 \)

4. Let \((x|1))\) be complex inner product space, and let \( \theta : X \rightarrow X \) be any linear map such that

\( (\theta v|v) = 0 \ \forall v \in X \) then \( \theta = 0 \), the zero map.

**Solution.** For all \( x, y \in X \) and all \( \alpha \in \mathbb{C} \) we have,

\[ 0 = (\theta(ax + y)|\alpha x + y) \]
\[ = (\theta(ax) + \theta y|\alpha x + y) \]
\[ = (\theta(ax)|\alpha x) + (\theta(ax)|y) + (\theta(x)|\alpha x) + (\theta y|y) \]
\[ = (\alpha(x)|y) + (\theta y|\alpha x) \]
\[ = \alpha(\theta(x)|y) + \alpha(\theta y|x) \]

... (1)

Put first \( \alpha = 1 \) and then \( \alpha = i \) in the equation (1), we get

\[ (\theta(x)|y) + (\theta(y)|x) = 0 \] ... (2)
\[ -i(\theta(x)|y) + i(\theta(y)|x) = 0 \ \forall x, y \in X \] ... (3)

Applying \( i(2) + (3) \), we get

\[ 2i(\theta y|x) = 0, 2i \neq 0 \Rightarrow (\theta y|x) = 0 \ \forall x, y \in X \]
\[ \Rightarrow \theta(y) = 0 \ \forall y \in X \]
\[ (x_0, y) = 0 \ \forall y \in X \text{ iff } x_0 = 0 \]
\[ \Rightarrow \theta = 0, \text{ the zero map.} \]

**Note.** This is not the case when \((X, (1))\) is a real inner product space, for instance let \( \theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) rotate each \( x \in \mathbb{R}^2 \) by \( 90^\circ \).

5. Let \( V \) be a complex inner product space and let \( T \in L(V) \). Then \( T \) is self adjoint iff
\[(Tv|v) \in \mathbb{R} \ \forall v \in V \ [T \text{ is self adjoint if } T^* = T].\]

**Proof.** Let \( v \in V. \) Then,
\[
(Tv|v) - (Tv|v) = (Tv|v) - (vTv)
\]
\[
= (Tv|v) - (T^*v|v)
\]
\[
= ((T - T^*)v|v)
\]

If \((Tv|v) \in \mathbb{R} \ \forall v \in V \) then the left hand side of above equation becomes 0. So,
\[
((T - T^*)v|v) = 0 \ \forall \ v \in V
\]
\[
\Rightarrow (T - T^*)v = 0 \ \forall \ v \in V
\]
\[
\Rightarrow T - T^* = 0
\]
\[
\Rightarrow T = T^*
\]
and hence \( T \) is self adjoint.

Conversely, if \( T \) is self adjoint then the right hand side of above equation becomes 0. So \((Tv|v) = (Tv|v)\) for every \( v \in V, \) this implies that \((Tv|v) \in \mathbb{R} \) for every \( v \in V \) as desired.