Self Evaluation Test

1. Which of the following functions $\beta : R^2 \times R^2 \rightarrow R$ are bilinear forms?

(a) $\beta(x, y) = 1, x = (x_1, x_2), y = (y_1, y_2)$

(b) $\beta(x, y) = (x_1 + y_1)^2 - (x_1 - y_1)^2$

(c) $\beta(x, y) = 1, x = (x_1y_2) - (x_2y_1)$

($R^2$ is a $R$-module, $R \sim$ set of real no’s)

Solution (a) $\beta(x, z) = 1, \beta(y, z) = 1, \beta(x + y, z) = 1 \neq \beta(x, z) + \beta(y, z)$

$\beta(x, y) = 1$ not a bilinear form.

(b) $\beta(x, y) = x_1^2 + y_1^2 + 2x_1y_1 - x_2^2 - y_2^2 + 2x_1y_1$

$\beta(x, y) = 4x_1y_1$

$\beta(\lambda x, \mu y) = \beta((\lambda x_1, \lambda x_2), (\mu y_1, \mu y_2))$

$= \beta(\lambda(x_1, x_2), (\mu y_1, \mu y_2))$

$= 4\lambda x_1 \mu y_1$

$= \lambda \mu \beta(x, y)$.

$\beta(x + \mu, y) = \beta((x_1, x_2) + (\mu_1, \mu_2), (y_1, y_2))$

$= \beta((x_1 + \mu_1, x_2 + \mu_2), (y_1, y_2))$

$= 4(x_1 + \mu_1)y_1$

$= 4x_1y_1 + \mu_1y_1$

$= \beta(x, y) + \beta(\mu, y)$

$\Rightarrow \beta$ is a bilinear form.

(c) $\beta(x, y) = x_1y_2 - x_2y_1$

$\beta(\lambda x, \mu y) = \beta((\lambda x_1, \lambda x_2), (\mu y_1, \mu y_2))$

$= \lambda x_1 \mu y_2 - \lambda x_2 \mu y_1$

$= \lambda \mu x_1y_2 - \lambda \mu x_2y_1$

$= \lambda \mu (x_1y_2 - x_2y_1)$

$= \lambda \mu \beta(x, y)$.

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\[\beta(x + \mu, y) = \beta((x_1 + \mu_1, x_2 + \mu_2), (y_1, y_2))\]
\[= (x_1 + \mu_1)y_2 - (x_2 + \mu_2)y_1\]
\[= x_1y_2 + \mu_1y_2 - x_2y_1 - \mu_2y_1\]
\[= x_1y_2 - x_2y_1 + \mu_1y_2 - \mu_2y_1\]
\[= \beta(x, y) + \beta(\mu, y)\]
\[\Rightarrow \beta \text{ is a bilinear form.}\]

2. The following expressions define quadratic forms \(Q\) on \(\mathbb{R}^2\). Find the symmetric bilinear form \(\beta\) corresponding to each \(Q\).

(a) \(ax_1^2\)
(b) \(3x_1x_2 - x_2^2\)

Solution.

\[Q(x) = ax_1^2\]
\[= \beta(x, x) \text{ where } x = (x_1, x_2)\]
\[\beta(x, y) = \frac{1}{4}[Q(x + y) - Q(x - y)]\]
\[Q(x + y) = Q((x_1 + y_1), (x_2 + y_2))\]
\[= a(x_1 + y_1)^2\]
\[Q(x - y) = a(x_1 - y_1)^2\]
\[\beta(x, y) = \frac{1}{4}[a(x_1 + y_1)^2 - a(x_1 - y_1)^2]\]
\[= \frac{4a}{4}x_1y_1\]
\[= ax_1y_1\]

(b)

\[Q(x) = 3x_1x_2 - x_2^2\]
\[Q((x_1 + y_1), (x_2 + y_2)) = Q(x + y)\]
\[= 3(x_1 + y_1)(x_2 + y_2) - (x_2 + y_2)^2\]
\[Q(x - y) = Q((x_1 - y_1), (x_2 - y_2))\]
\[= 3(x_1 - y_1)(x_2 - y_2) - (x_2 - y_2)^2\]
\[\beta(x, y) = \frac{1}{4}[Q(x + y) - Q(x - y)]\]
\[= \frac{1}{4}[3(x_1 + y_1)(x_2 + y_2) - (x_2 + y_2)^2 - 3(x_1 - y_1)(x_2 - y_2) + (x_2 - y_2)^2]\]
\[= \frac{1}{4}[6x_1y_2 + 6y_1x_2 - 4x_2y_2]\]
\[= \frac{1}{2}[3x_1y_2 + 3y_1x_2 - 2x_2y_2]\]
Let $Q$ be a quadratic form associated with symmetric bilinear form $\beta$. Verify the polar identity

$$\beta(x, y) = \frac{1}{2} [Q(x + y) - Q(x) - Q(y)].$$

**Solution.**

\[
\begin{align*}
\frac{1}{2} [Q(x + y) - Q(x) - Q(y)] &= \frac{1}{2} \left[ \beta(x + y, x + y) - \beta(x, x) - \beta(y, y) \right] \\
&= \frac{1}{2} \left[ \beta(x, x) + \beta(x, y) + \beta(y, x) + \beta(y, y) - \beta(x, y) - \beta(y, y) \right] \\
&= \frac{1}{2} \times 2 \beta(x, y) \\
&= \beta(x, y)
\end{align*}
\]

4. Every bilinear form on a vector space $X$ over a field $F$ can be uniquely expressed as the sum of a symmetric and skew-symmetric bilinear forms.

**Solution.** We know that every vector space over a field is also a module.

Let $\beta$ be a bilinear form on a vector space $X$ over $F$.

Let $g(x, y) = \frac{1}{2} [\beta(x, y) + \beta(y, x)]$

\[
\begin{align*}
g(x, y) &= \frac{1}{2} [\beta(x, y) - \beta(y, x)] \quad \forall \ x, y \in X
\end{align*}
\]

Therefore $g$ and $h$ are also bilinear form on $X$.

\[
\begin{align*}
g(y, x) &= \frac{1}{2} [\beta(y, x) + \beta(x, y)] \\
&= g(x, y). \\
\Rightarrow g \text{ is symmetric.}
\end{align*}
\]

\[
\begin{align*}
h(y, x) &= \frac{1}{2} [\beta(y, x) - \beta(x, y)] \\
&= -\frac{1}{2} [\beta(x, y) - \beta(y, x)] \\
&= -h(x, y). \\
\Rightarrow h \text{ is skew-symmetric.}
\end{align*}
\]

\[
\begin{align*}
\beta(x, y) &= g(x, y) + h(x, y) \\
\Rightarrow \beta &= g + h.
\end{align*}
\]

Now suppose that $\beta = \beta_1 + \beta_2$ where $\beta_1$ is the symmetric bilinear form and $\beta_2$ is skew symmetric bilinear form.

\[
\begin{align*}
\beta(x, y) &= (\beta_1 + \beta_2)(x, y) \\
&= \beta_1(x, y) + \beta_2(x, y) \\
&= \beta_1(x, y) + \beta_2(y, x) \\
&= \beta_1(x, y) - \beta_2(x, y)
\end{align*}
\]

Adding (1) and (2), we get
\[ 2 \beta_1(x, y) = \beta(x, y) + \beta(y, x) \]
\[ \beta_1(x, y) = \frac{1}{2} [\beta(x, y) + \beta(y, x)] \]
\[ = g(x, y) \]
\[ \Rightarrow \beta_1 = g \]

Similarly it can be proved that \( \beta_2 = h \).

5. Can a sesquilinear is a bilinear form.

**Solution.** Only zero form is both bilinear and sesquilinear form. Non zero sesquilinear form can not be a bilinear form.

Suppose \( \beta \) is a sesquilinear form and bilinear form.
\[
\beta(x, \lambda y) = \bar{\lambda} \beta(x, y) \\
\beta(x, \lambda y) = \lambda \beta(x, y) \\
(\bar{\lambda} - \lambda) \beta(x, y) = 0 \quad \text{(In general } \bar{\lambda} - \lambda \neq 0) \\
\Rightarrow \beta(x, y) = 0 \quad \forall x, y \in X \\
\Rightarrow \beta \text{ is zero form.}