Self Evaluation Test

1. Let \( \lambda \) be an eigen value of a linear operator \( T \) on a vector space \( V(\mathbb{F}) \). Let \( V_\lambda \) denote the set of all eigen vectors of \( T \) corresponding to eigen value \( \lambda \). Prove that \( V_\lambda \) is a subspace of \( V(\mathbb{F}) \).

**Solution.** Here \( V_\lambda = \{ v \in V \mid v \) is an eigen vector of \( T \} \).

\[ V_\lambda = \{ v \in V \mid T(v) = \lambda v \}. \]

Given is that \( \lambda \) be an eigen value of \( T \).

\[ \exists \ a \ non \ zero \ vector \ v' \ such \ that \ T(v') = \lambda v' \ so \ that \ v' \in V_\lambda \Rightarrow V_\lambda \neq \phi \]

i.e. \( V_\lambda \) is non-empty set.

let \( v_1, v_2 \in V_\lambda \) and \( \alpha, \beta \in \mathbb{F} \)

Since \( v_1, v_2 \in V_\lambda \Rightarrow Tv_1 = \lambda v_1 \) and \( Tv_2 = \lambda v_2 \)

Now \[ T(\alpha v_1 + \beta v_2) = T(\alpha v_1) + T(\beta v_2) \]

\[ = \alpha T(v_1) + \beta T(v_2) \]

\[ = \alpha \lambda v_1 + \beta \lambda v_2 \]

\[ = \lambda (\alpha v_1 + \beta v_2) \]

\[ \Rightarrow \alpha v_1 + \beta v_2 \) is an eigen vector corresponding to eigen value \( \lambda \)

\[ \Rightarrow \alpha v_1 + \beta v_2 \in V_\lambda \]

Hence \( V_\lambda \) is a subspace of \( V \).

2. Prove that the non zero eigen vectors corresponding to distinct eigen values of a linear operator are linearly independent.

**Solution.** let \( v_1, v_2, \ldots , v_m \) be \( m \) non-zero eigen vectors of a linear operator \( T : V \to V \) corresponding to distinct eigen values \( \lambda_1, \lambda_2, \ldots , \lambda_m \) respectively.

\[ \Rightarrow T(v_1) = \lambda_1 v_1, \ T(v_2) = \lambda_2 v_2, \ldots , \ T(v_m) = \lambda_m v_m \]

We want to show that \( v_1, v_2, \ldots , v_m \) are L.I. vectors. We shall prove this result by induction on \( m \).

**Step 1.** Let \( m = 1 \)

Then \( v_1 \) is L.I. since \( v_1 \) is a non-zero vector.

\[ \therefore \ the \ result \ is \ true \ for \ m = 1. \]
Step II. Assume the result is true for the number of vectors less than $m$.

Step III. Now, we shall show the result is true for $m$ vectors.

Let 
\[ a_1 v_1 + a_2 v_2 + \ldots + a_m v_m = 0 \quad (2) \]

\[ \Rightarrow T(a_1 v_1 + a_2 v_2 + \ldots + a_m v_m) = T(0) \]

\[ \Rightarrow a_1 T(v_1) + a_2 T(v_2) + \ldots + a_m T(v_m) = 0 \quad [\text{Since } T \text{ is a L.T.}] \]

\[ \Rightarrow a_1 (\lambda_1 v_1) + a_2 (\lambda_2 v_2) + \ldots + a_m (\lambda_m v_m) = 0 \quad [\text{Using (1)}] \]

\[ \Rightarrow a_1 v_1 + a_2 v_2 + \ldots + a_m (\lambda_v v_m) = 0 \quad (3) \]

Multiplying (2) on both sides by $\lambda_m$, we get

\[ a_1 (\lambda_1 v_1) + a_2 (\lambda_2 v_2) + \ldots + a_m (\lambda_m v_m) = 0 \quad (4) \]

\[ \therefore \text{eq}(3)-\text{eq}(4) \text{ gives} \]

\[ a_1 (\lambda_1 - \lambda_m) v_1 + a_2 (\lambda_2 - \lambda_m) v_2 + \ldots + a_{m-1} (\lambda_{m-1} - \lambda_m) v_{m-1} = 0 \]

\[ \Rightarrow a_1 (\lambda_1 - \lambda_m) = 0, \ a_2 (\lambda_2 - \lambda_m) = 0, \ldots, a_{m-1} (\lambda_{m-1} - \lambda_m) = 0 \]

\[ (\because \ v_1, v_2, \ldots, v_{m-1} \text{ are L.I. because of Step II}) \]

\[ \Rightarrow a_1 = 0, a_2 = 0, \ldots, a_{m-1} = 0 \]

\[ (\because \ \lambda_i - \lambda_m \neq 0 \text{ for } 1 \leq i \leq m-1 \text{ as } \lambda_i \text{ are distinct}) \]

Putting these in (2), we get

\[ a_m v_m = 0 \]

\[ \Rightarrow a_m = 0 \quad [\because v_m \neq 0] \]

Thus we have $a_1 = a_2 = \ldots = a_m = 0$

\[ \therefore \text{the vectors } v_1, v_2, \ldots, v_m \text{ are L.I.} \]

Hence the result

3. Let $\lambda$ be an eigen value of an invertible operator $T$ on a vector space $V(F)$. Prove that $\lambda^{-1}$ is an eigen value of $T^{-1}$

Solution. Given $T$ be invertible operator.

\[ \Rightarrow T \text{ is non-singular.} \]

\[ \Rightarrow \exists \text{ an eigen value } \lambda \neq 0. \]

\[ \Rightarrow \lambda^{-1} \text{ exists.} \]

Since $\lambda$ is an eigen value of $T$, therefore there exists a non zero vector $v \in V$ such that

\[ T(v) = \lambda v \]

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operating $T^{-1}$ on both sides

\[ T^{-1}(T(v)) = T^{-1}(\lambda v) \]

\[ v = \lambda T^{-1}(v) \]

\[ \frac{1}{\lambda} v = T^{-1}(v) \]

or $T^{-1}(v) = \frac{1}{\lambda} v = \lambda^{-1}(v)$

\[ \lambda^{-1} \text{ is an eigen value of } T^{-1} \]

Hence the result.

4. Let $V$ be vector space of all real valued continuous functions. Let $T$ be a L.O. on $V$ such that

\[ T(f(x)) = \int_{0}^{x} f(t) \, dt \]

Show $T$ has no eigen values.

**Solution.** If $\lambda$ is an eigen value of $T$, then by definition of eigen value, $\exists$ a non zero $f(x) \in V$ such that

\[ T(f(x)) = \lambda f(x) \]

\[ \Rightarrow \int_{0}^{x} f(t) \, dt = \lambda f(x) \tag{1} \]

Differentiating both sides we get $f(x) = \lambda f'(x)$

\[ \Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{\lambda} \]

Integrating, we get

\[ \log f(x) = \frac{x}{\lambda} + C, \text{ } C \text{ is constant of integration} \]

\[ \Rightarrow f(x) = e^{\frac{x}{\lambda} + C} = e^{C} e^{\frac{x}{\lambda}} = ae^{\frac{x}{\lambda}} \text{ say} \]

\[ \therefore f(0) = ae^{0} = a \]

so that $f(x) = f(0)e^{\frac{x}{\lambda}} \tag{ii}$

changing variable $x$ by $t$ we have

\[ f(t) = f(0) e^{\frac{t}{\lambda}} \]

integrating both sides from 0 to $x$ we get

\[ \int_{0}^{x} f(t) \, dt = f(0) \int_{0}^{x} e^{\frac{t}{\lambda}} \, dt \]

\[ \lambda f(x) = f(0) \left[ \frac{e^{\frac{x}{\lambda}}} {\frac{x}{\lambda}} \right]_{0}^{x} \text{ (using (i) for L.H.S)} \]

\[ \lambda f(0)e^{\frac{x}{\lambda}} = f(0)\lambda(e^{\frac{x}{\lambda}} - 1) \text{ (using (ii) for L.H.S)} \]

\[ e^{\frac{x}{\lambda}} = e^{\frac{x}{\lambda}} - 1 \]

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\[
\Rightarrow 0 = -1 \text{ which is wrong.}
\]
So that \( T \) has no eigen values.

5. Let \( T : V \to V \) be a Linear operator on a finite dimensional vector space \( V(\mathbb{F}) \). Prove that the number of eigen values of \( T \) cannot exceed the dimension of vector space \( V(\mathbb{F}) \).

**Solution.** Given \( V \) is a finite dimensional vector space over \( \mathbb{F} \).

Let us assume \( \dim V = n \).

Now \( \lambda \) is an eigen value of \( T \) iff \( \det (\lambda I - T) = 0 \)
i.e., the eigen values of \( T \) are the roots of equation
\[
\det(xI - T) = 0 \tag{1}
\]
Since \( \dim V = n \), so any matrix representation of \( T \) is of order \( n \times n \).

\[\Rightarrow\] the matrix representation of \( xI - T \) is also of order \( n \times n \).

But the eigen values of \( T \) are roots of this polynomial \([\text{because of (i)}]\)

\[\therefore\] number of eigen values cannot exceed the degree \( n \) of the polynomial \( \det(xI - T) \).

Hence the number of eigen values of \( T \) cannot exceed the \( \dim V \).